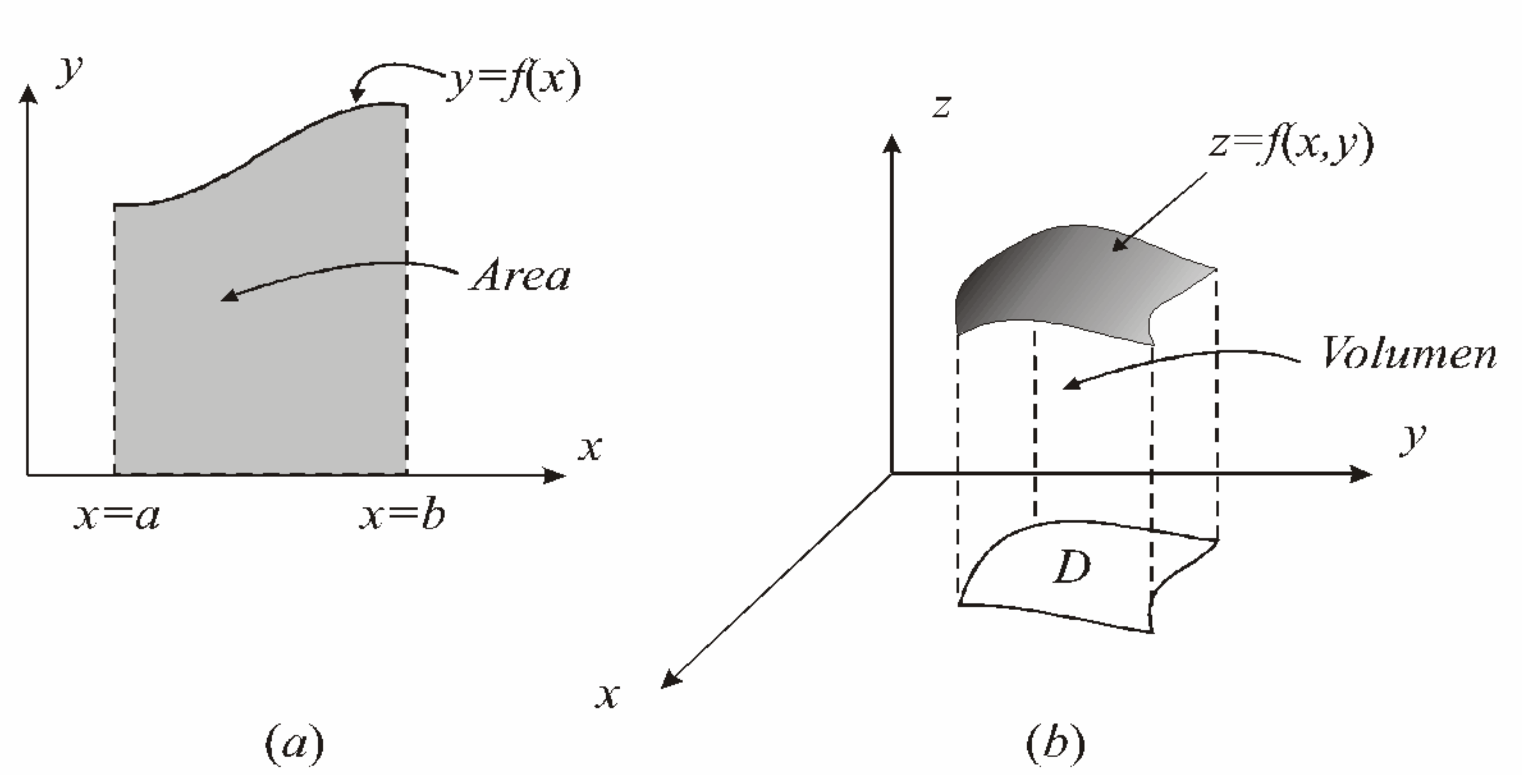
CAPÍTULO 5. INTEGRALES MÚLTIPLES

- **5.1 INTEGRALES DOBLES**
- **5.2 INTEGRALES TRIPLES**
- **5.3 APLICACIONES**

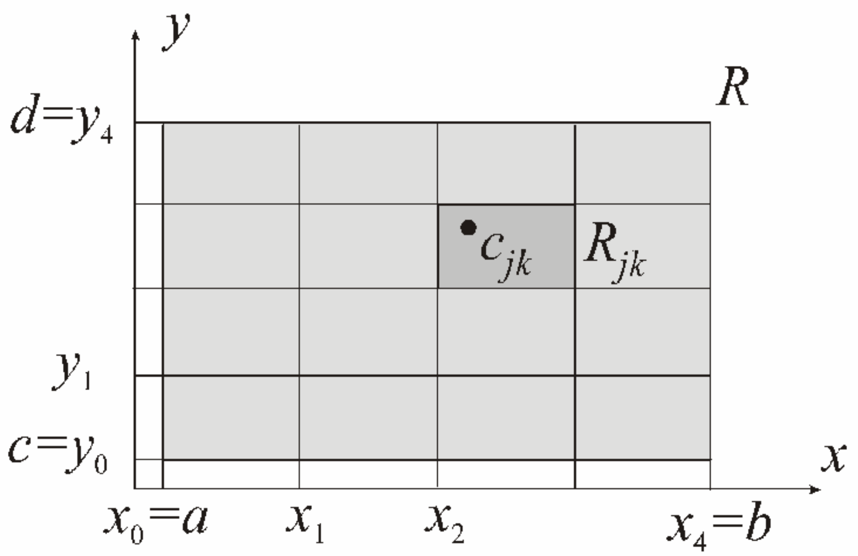
5.1 INTEGRALES DOBLES INTRODUCCIÓN



(a) Área bajo una curva. (b) Volumen bajo una superficie.

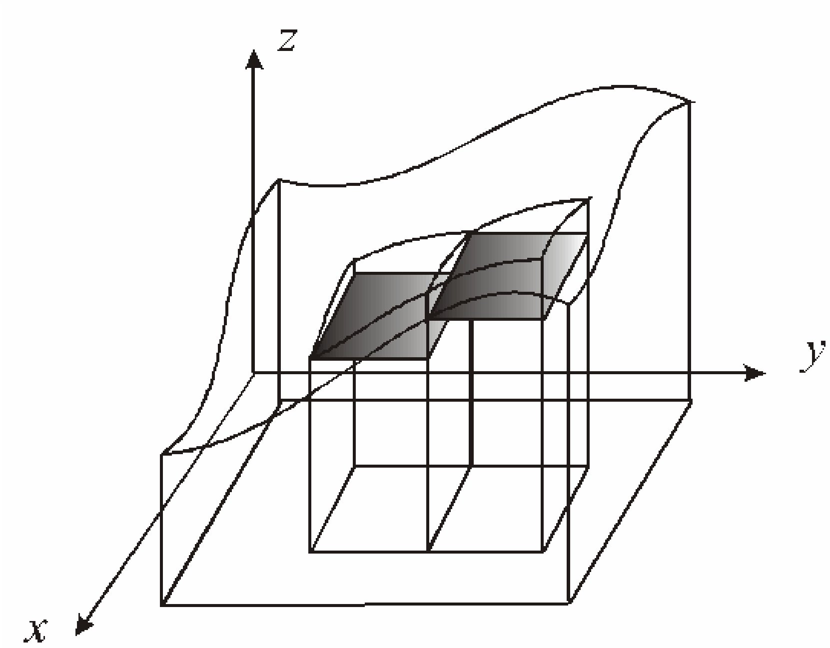
5.1 INTEGRALES DOBLES SOBRE RECTÁNGULOS

$$S_n = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f\left(c_{jk}\right) \Delta x \Delta y$$

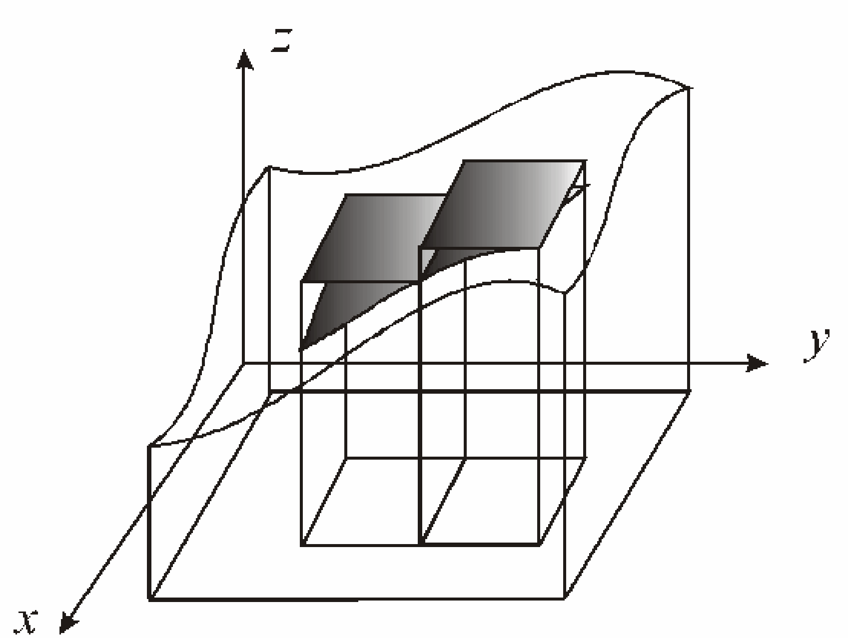


$$\lim_{n\to\infty} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} f\left(c_{jk}\right) \Delta x \Delta y = \iint_R f\left(x, y\right) dx dy$$

5.1 INTEGRALES DOBLES INTERPRETACIÓN como VOLUMEN



Volumen de un sólido inscrito.



Volumen de un sólido circunscrito.

5.1 INTEGRALES DOBLES PROPIEDADES

a) Linealidad.

$$\iint_R (f+g)\,dxdy = \iint_R f\,dxdy + \iint_R g\,dxdy;$$

$$\iint_R \alpha f\,dxdy = \alpha \iint_R f\,dxdy$$

b) Monotonía.

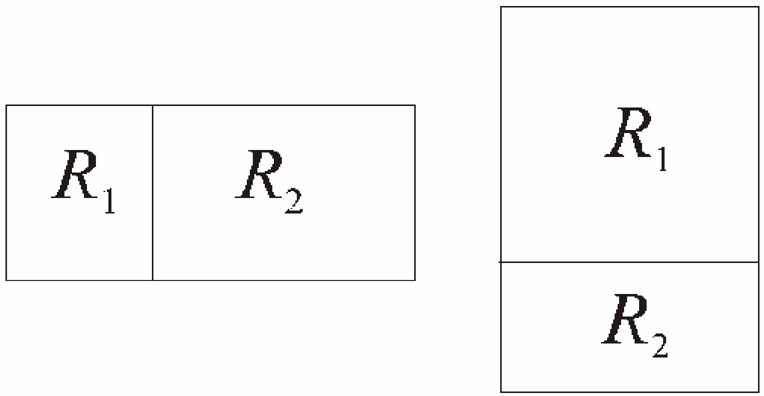
$f(x,y) \leq g(x,y)$ entonces $\displaystyle\iint_R f\,dxdy \leq \iint_R g\,dxdy$

$$\left|\iint_R f\,dxdy\right| \leq \iint_R |f|\,dxdy$$

5.1 INTEGRALES DOBLES PROPIEDADES

c) Aditividad respecto a rectángulos.

$$\iint_R f\,dx\,dy = \iint_{R_1} f\,dx\,dy + \iint_{R_2} f\,dx\,dy$$



5.1 INTEGRALES DOBLES PROPIEDADES

Valor medio

$$f_R = \frac{1}{\acute{a}rea(R)} \iint_R f(x,y)\,dxdy$$

Teorema del valor medio para integrales dobles

Supongamos que $f : R \longrightarrow \mathbb{R}$ *es continua.*

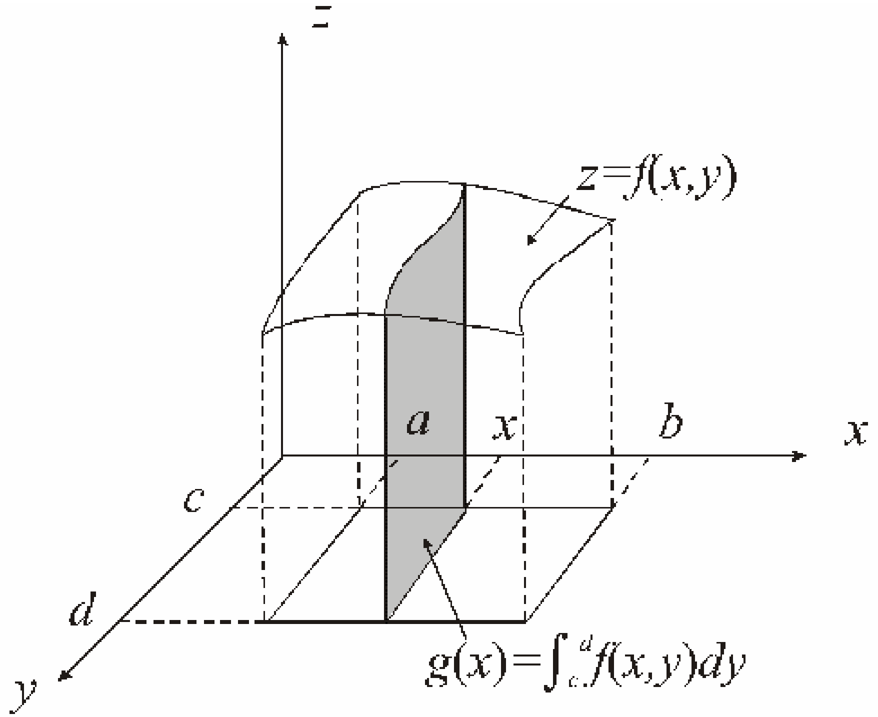
Entonces para algún punto (x_0, y_0) *en* R*, tenemos*

$$\iint_R f(x,y)\,dxdy = f(x_0, y_0)\,A(R)$$

5.1 INTEGRALES DOBLES INTEGRALES ITERADAS

$$\iint f(x,y)\,dxdy = \int \left[\int f(x,y)\,dx \right] dy;$$

$$\iint f(x,y)\,dydx = \int \left[\int f(x,y)\,dy \right] dx$$



Teorema de Fubini sobre un rectángulo

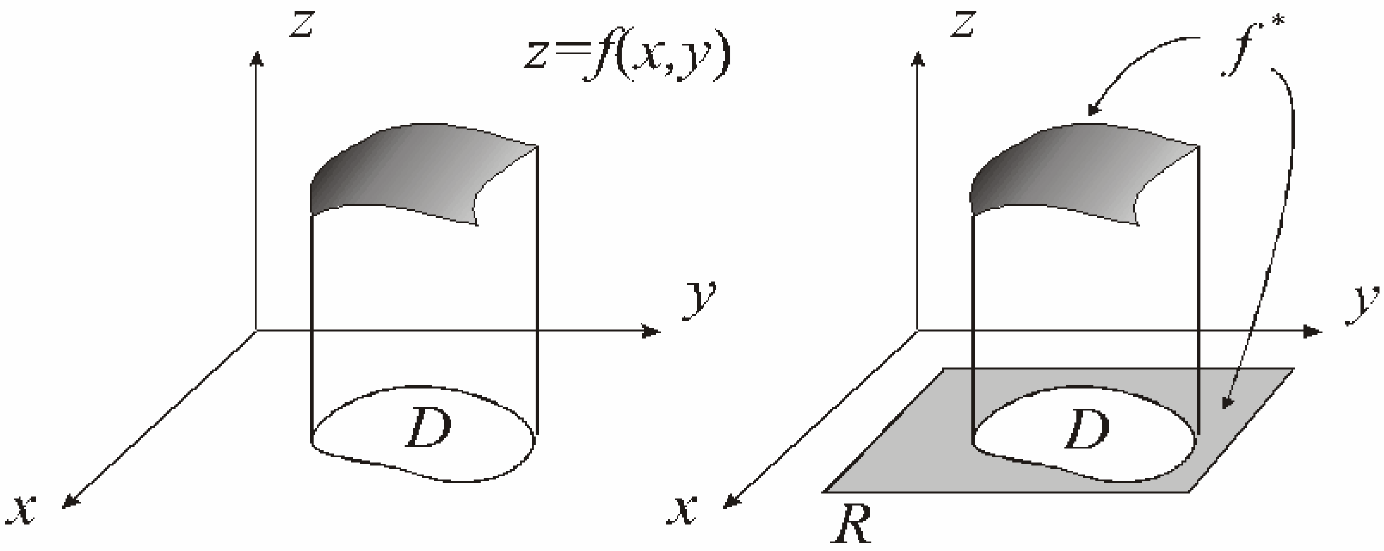
Supongamos que $f(x,y)$ es una función continua sobre un rectángulo R

$$\iint_R f(x,y)\,dxdy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

5.1 INTEGRALES DOBLES SOBRE REGIONES GENERALES

$$\iint_D f\,dxdy = \iint_R f \cdot \chi_D\,dxdy = \iint_R f^*\,dxdy$$

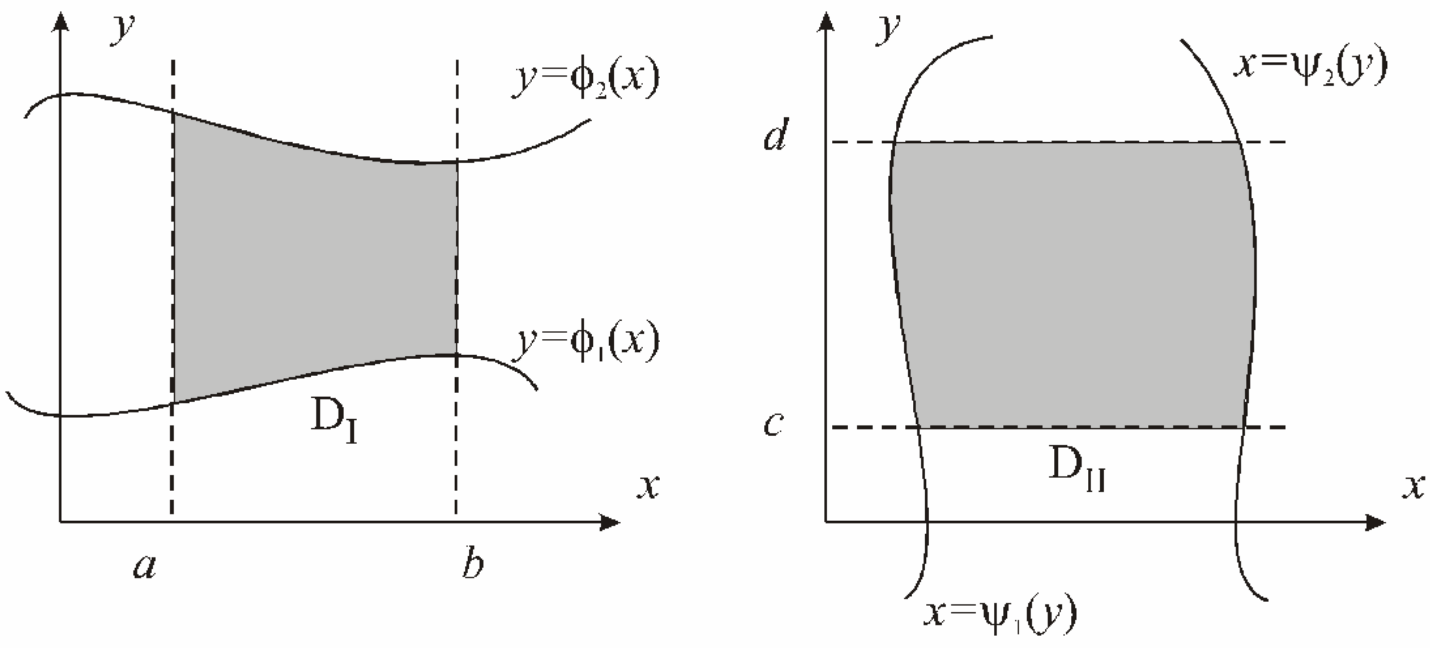
$$f^* = f \cdot \chi_D = \begin{cases} f(x,y) & \text{si} \quad (x,y) \in D \\ 0 & \text{si} \quad (x,y) \in R - D \end{cases}$$



5.1 INTEGRALES DOBLES
REGIONES de INTEGRACIÓN

REGIONES DE TIPO (I) $D_I = \{(x, y) / a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\}$

REGIONES DEL TIPO (II) $D_{II} = \{(x, y) / \psi_1(y) \leq x \leq \psi_2(y), c \leq y \leq d\}$



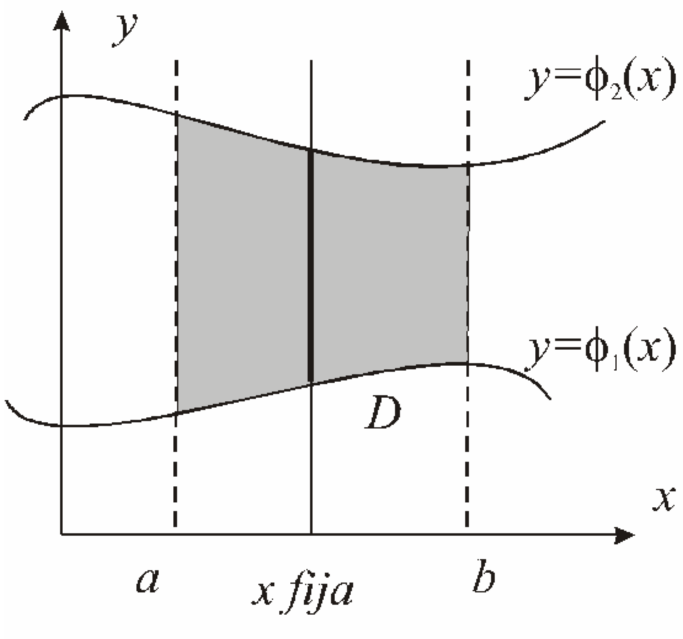
(a) (b)

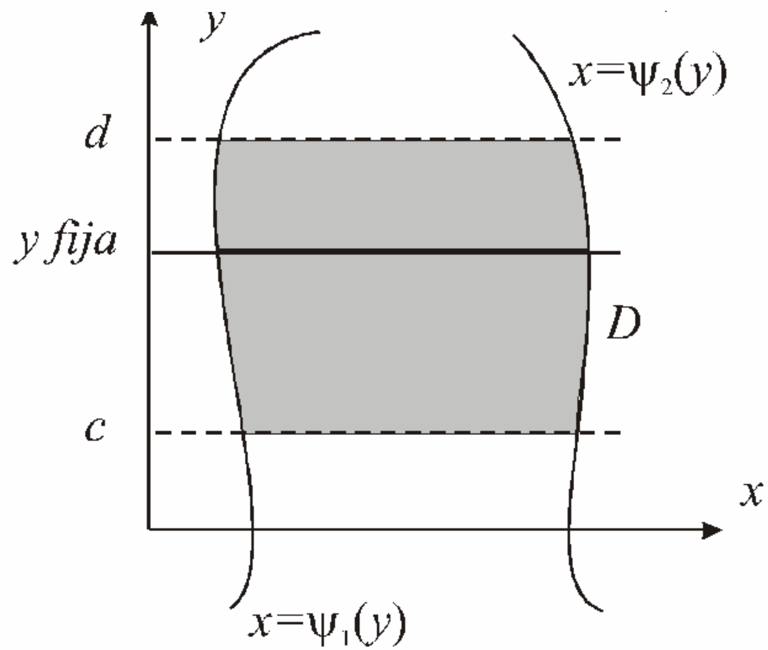
5.1 INTEGRALES DOBLES
T^MA FUBINI para REGIONES de INTEGRACIÓN

Supongamos que f *es continua en* D

$$\iint_D f(x,y)\,dxdy = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right) dx$$

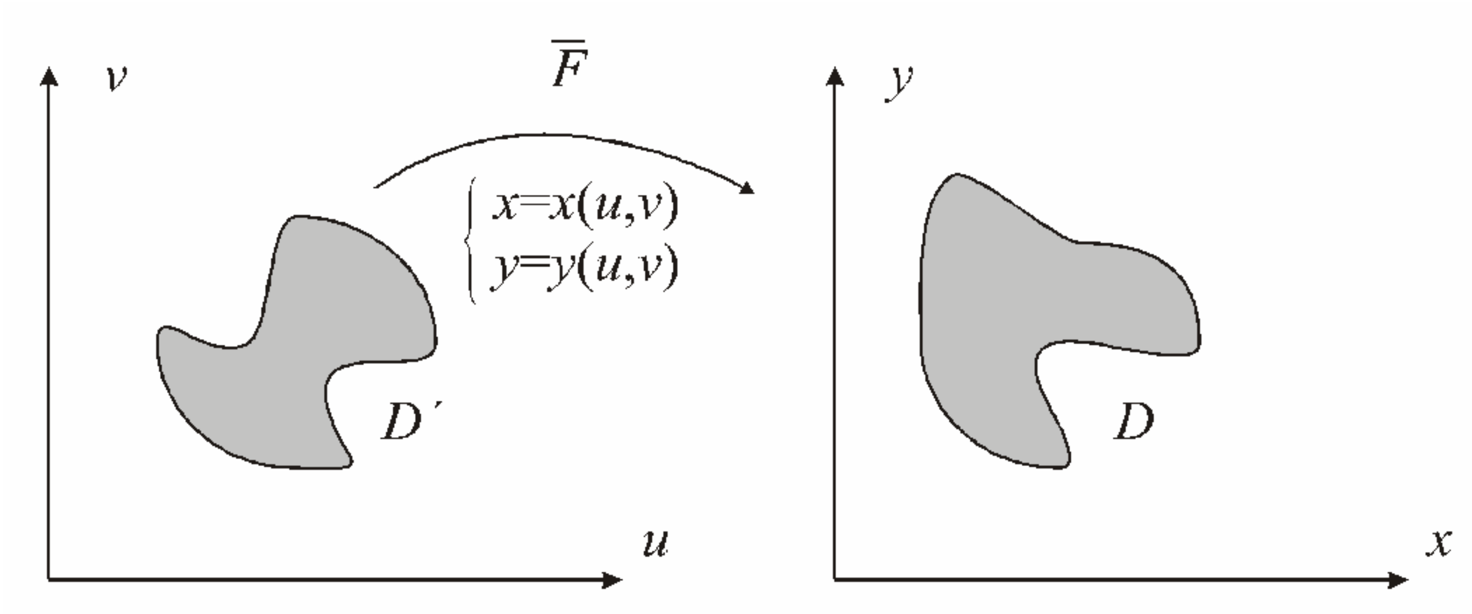
$$\iint_D f(x,y)\,dxdy = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right) dy$$





5.1 INTEGRALES DOBLES
CAMBIO de VARIABLES

$$
\begin{aligned}
\overline{F} \quad &: \quad D' \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\
\overline{F}(u,v) \quad &= \quad (x,y) = (x\,(u,v)\,, x(u,v))
\end{aligned}
$$

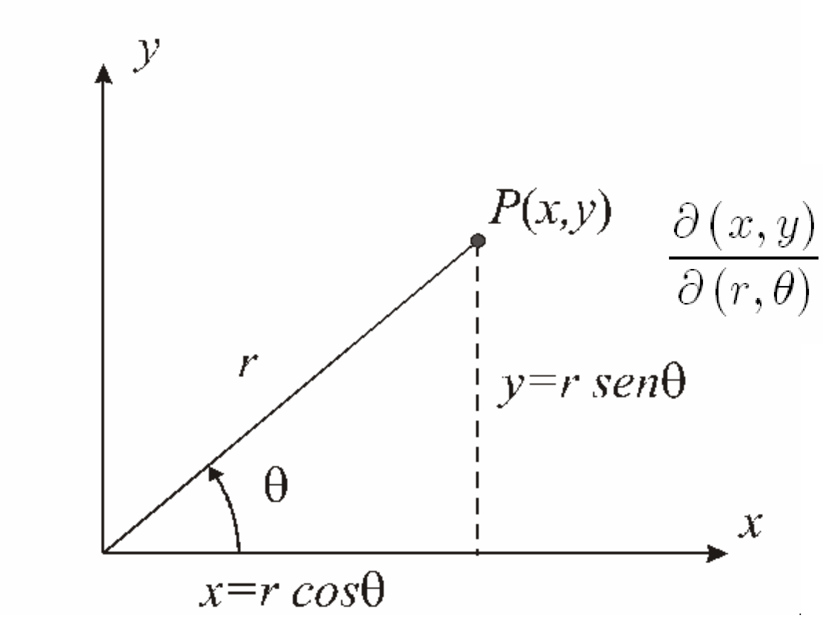


5.1 INTEGRALES DOBLES
CAMBIO de VARIABLES

$$\begin{aligned}\iint_D f(x,y)dxdy &= \iint_{D'} f\left(x\left(u,v\right),y(u,v)\right)\left|\det\left[\overline{F}'(u,v)\right]\right|dudv \\ &= \iint_{D'} f\left(x\left(u,v\right),y(u,v)\right)\left|J(u,v)\right|dudv\end{aligned}$$

$$\det\left[\overline{F}'(u,v)\right] = J(u,v) = \frac{\partial\left(x,y\right)}{\partial\left(u,v\right)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix}$$

5.1 INTEGRALES DOBLES CAMBIO a POLARES

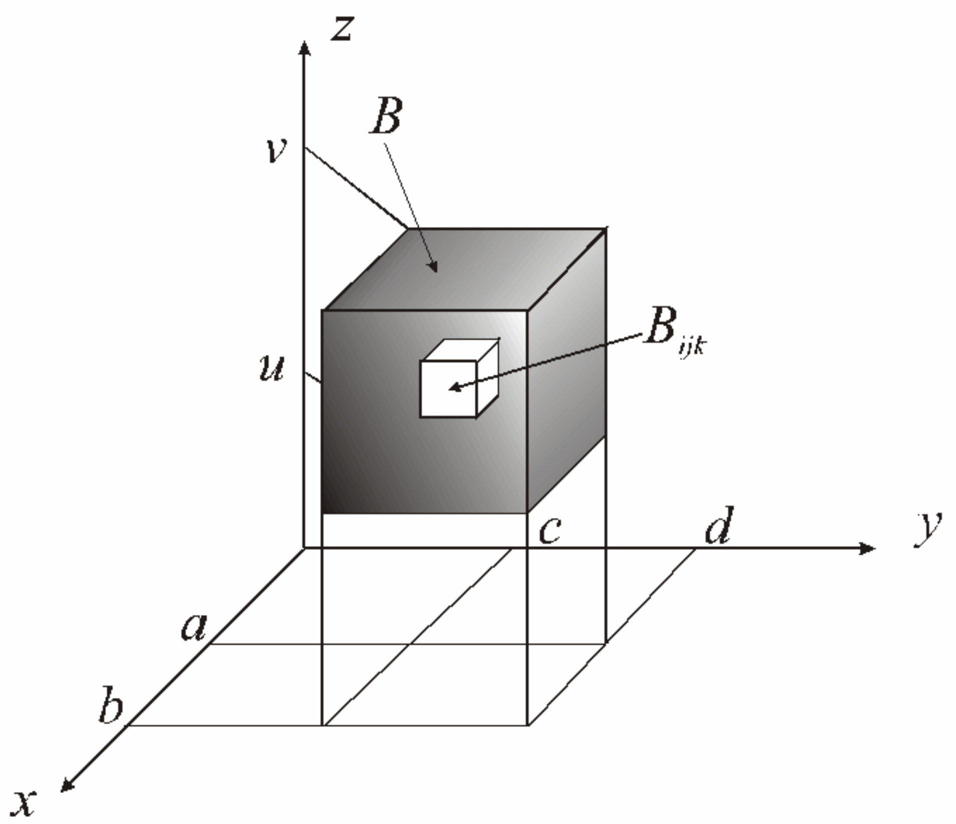


$$x = r\cos\theta, \quad y = r \operatorname{sen}\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det\begin{bmatrix} \cos\theta & -r\operatorname{sen}\theta \\ \operatorname{sen}\theta & r\cos\theta \end{bmatrix} = r$$

$$\iint_D f(x,y)\,dx\,dy = \iint_{D'} f(x(r,\theta), y(r,\theta)) \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dr\,d\theta = \iint_{D'} f(r\cos\theta, r\operatorname{sen}\theta)\, r\,dr\,d\theta$$

5.2 INTEGRALES TRIPLES SOBRE PARALELEPÍPEDOS



$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} f\left(c_{ijk}\right)\Delta x\Delta y\Delta z = \iiint_B f(x,y,z)\,dx\,dy\,dz$$

5.2 INTEGRALES TRIPLES
INTEGRALES ITERADAS

$$\int_u^v \left(\int_c^d \left(\int_a^b f(x,y,z)dx \right) dy \right) dz; \quad \int_u^v \left(\int_a^b \left(\int_c^d f(x,y,z)dy \right) dx \right) dz; \quad \text{etc.}$$

Teorema de Fubini sobre un paralelepípedo

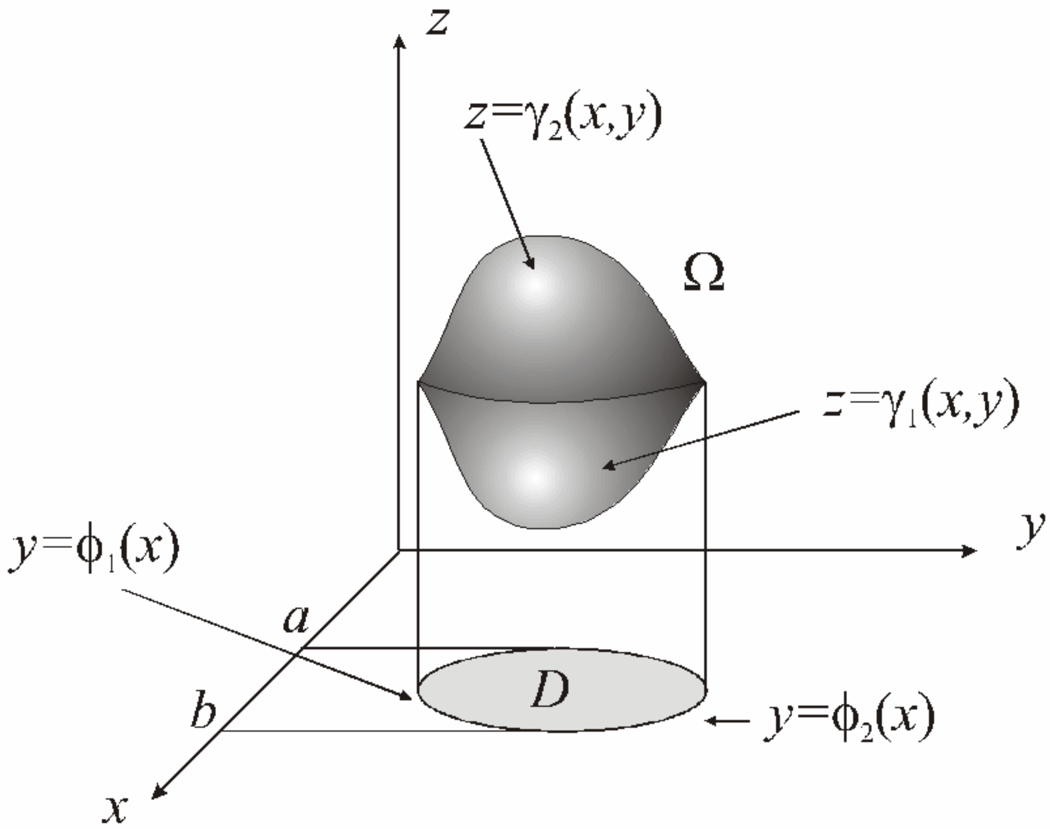
Si se cumple que $f(x,y,z)$ es continua sobre el paralelepípedo B

$$\iiint_B f(x,y,z)dxdydz = \int_u^v \left(\int_c^d \left(\int_a^b f(x,y,z)dx \right) dy \right) dz$$

5.2 INTEGRALES TRIPLES SOBRE REGIONES GENERALES

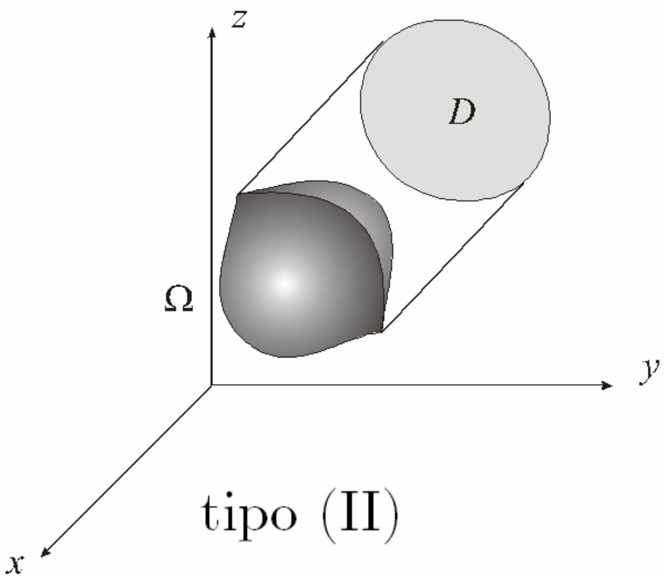
Regiones de integración en $\mathbb{R}^3$. tipo (I)

$$a \leq x \leq b, \qquad \phi_1(x) \leq y \leq \phi_2(x) \qquad \text{y} \qquad \gamma_1(x,y) \leq z \leq \gamma_2(x,y)$$

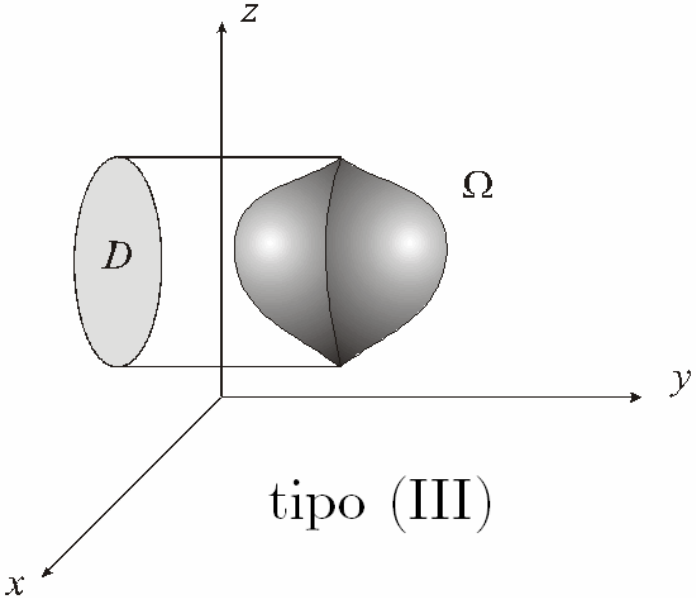


5.2 INTEGRALES TRIPLES SOBRE REGIONES GENERALES

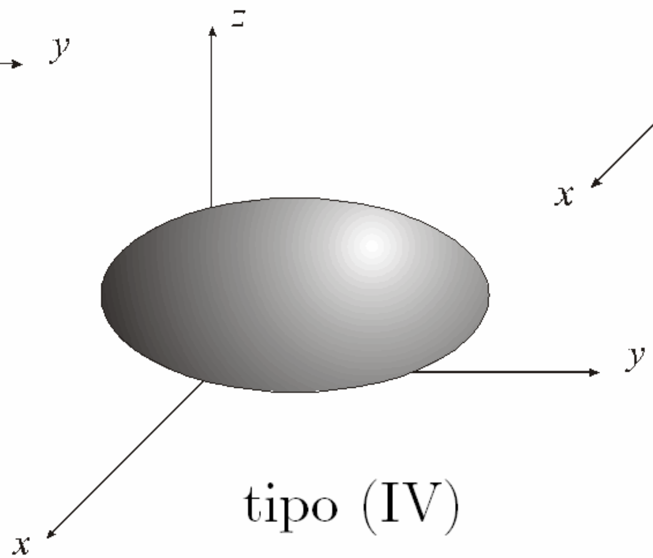
Regiones de integración en $\mathbb{R}^3$.



tipo (II)



tipo (III)



tipo (IV)

5.2 INTEGRALES TRIPLES
T^{MA} FUBINI para REGIONES de INTEGRACIÓN

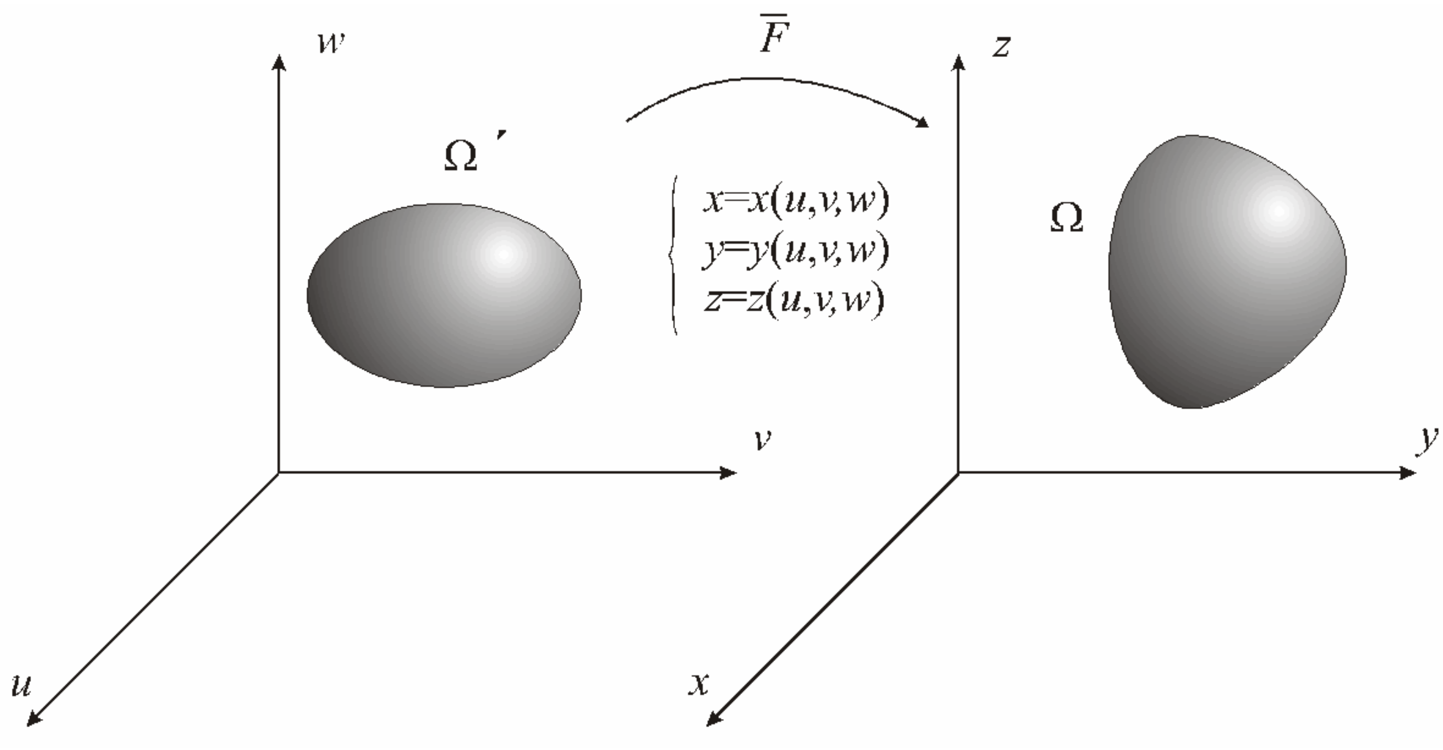
Si suponemos que Ω es del tipo (I), entonces

$$\iiint_\Omega f(x,y,z)\,dxdydz = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} \left(\int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\,dz \right) dy \right) dx =$$

$$= \iint_D \left[\int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\,dz \right] dydx$$

5.2 INTEGRALES TRIPLES CAMBIO de VARIABLES

$$\overline{F}(u,v,w) = (x,y,z) = (x(u,v,w), y(u,v,w), z(u,v,w))$$



5.2 INTEGRALES TRIPLES
CAMBIO de VARIABLES

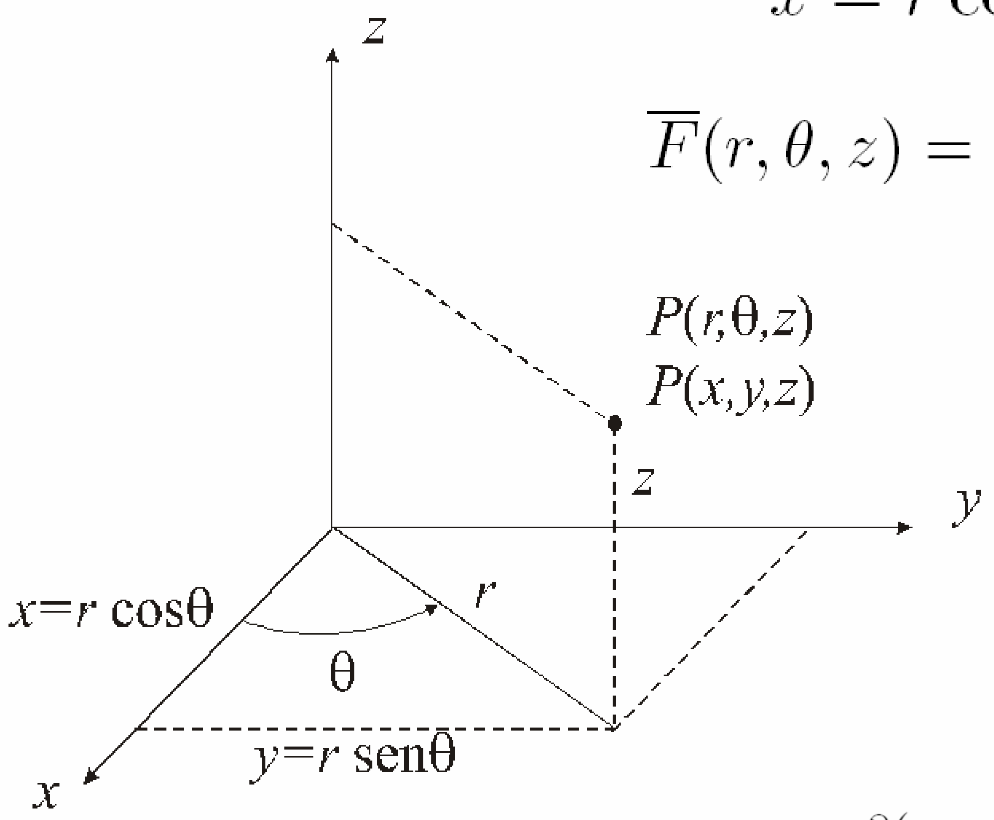
$$\iiint_{\Omega} f(x,y,z)dxdydz = \iiint_{\Omega'} f(\overline{F}(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw$$

$$J = \det \overline{F}'(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

5.2 INTEGRALES TRIPLES CAMBIO a CILÍNDRICAS

$$x = r\cos\theta, \quad y = r\,\text{sen}\,\theta, \quad z = z$$

$$\overline{F}(r,\theta,z) = (x,y,z) = (r\cos\theta, r\,\text{sen}\,\theta, z)$$

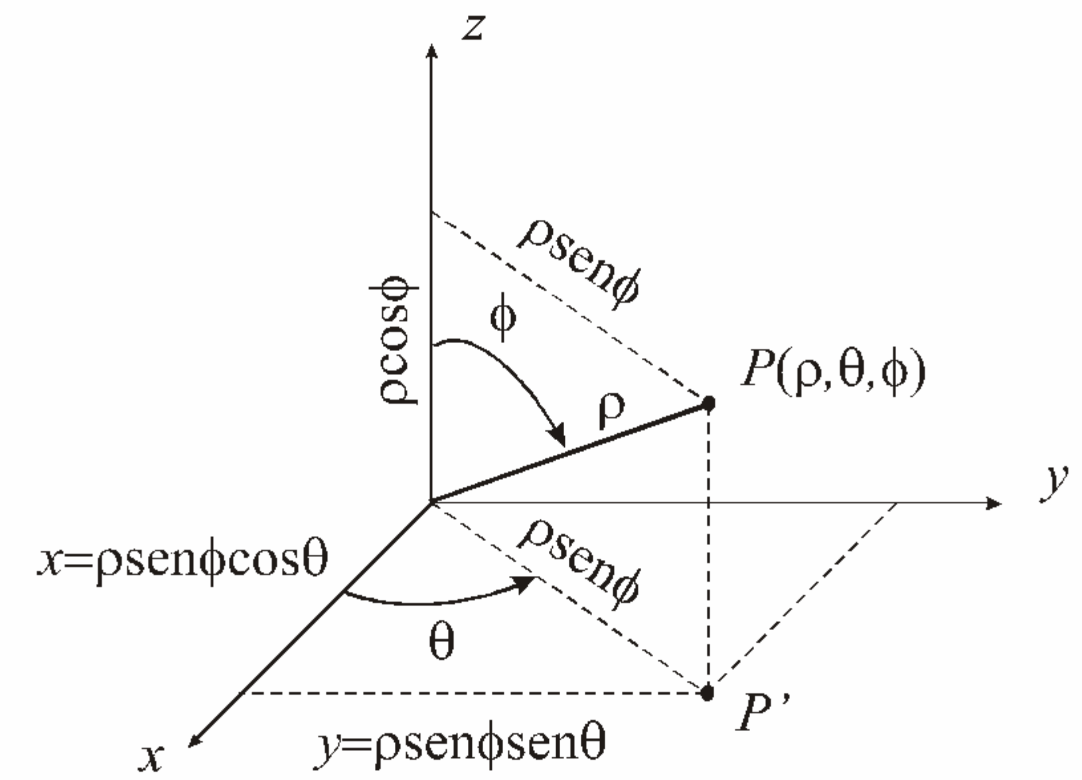


$$\det \overline{F}(r,\theta,z) = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \det\begin{bmatrix} \cos\theta & -r\,\text{sen}\,\theta & 0 \\ \text{sen}\,\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = r$$

5.2 INTEGRALES TRIPLES CAMBIO a ESFÉRICAS

$$x = \rho\cos\theta\,\mathrm{sen}\,\phi, \quad y = \rho\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi, \quad z = \rho\cos\phi$$

$$\overline{F}(\rho,\theta,\phi) = (x,y,z) = (\rho\cos\theta\,\mathrm{sen}\,\phi, \rho\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi, \rho\cos\phi)$$



$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = -\rho^2\,\mathrm{sen}\,\phi$$

5.3 APLICACIONES
VOLÚMENES y ÁREAS de figuras planas

$$\iiint_\Omega dxdydz = \text{volumen de } \Omega$$

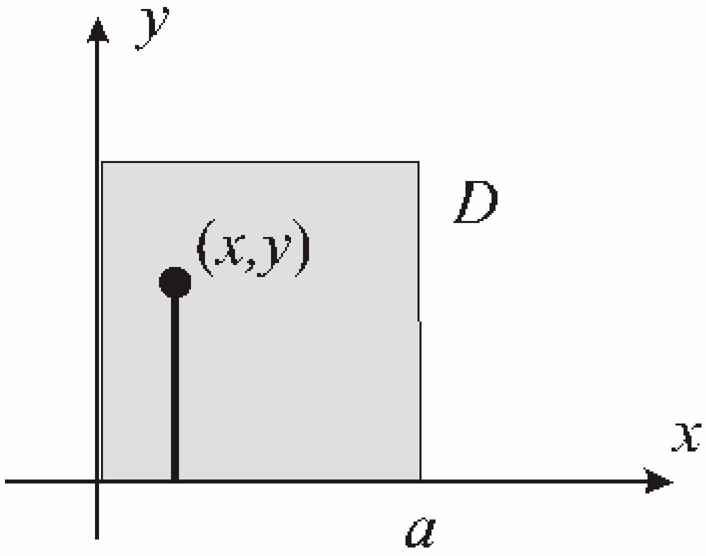
$$\iint_D dxdy = \text{área de } D$$

5.3 APLICACIONES

Centro de Masa y Momentos de figuras planas

$$m = \iint_D \mu(x,y)\,dx\,dy$$

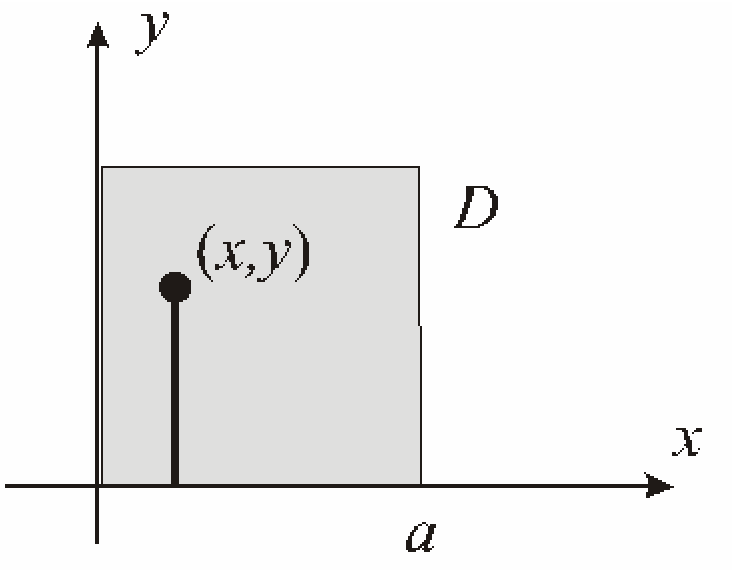
$$x_G = \frac{\iint_D x\mu(x,y)\,dx\,dy}{\iint_D \mu(x,y)\,dx\,dy}; \quad y_G = \frac{\iint_D y\mu(x,y)\,dx\,dy}{\iint_D \mu(x,y)\,dx\,dy}$$



5.3 APLICACIONES
Centro de Masa y Momentos de figuras planas

$$I_x = \iint_D y^2 \mu(x,y)\,dxdy; \quad I_y = \iint_D x^2 \mu(x,y)\,dxdy$$

$$I_0 = \iint_D \left(x^2 + y^2\right) \mu(x,y)\,dxdy$$



5.3 APLICACIONES

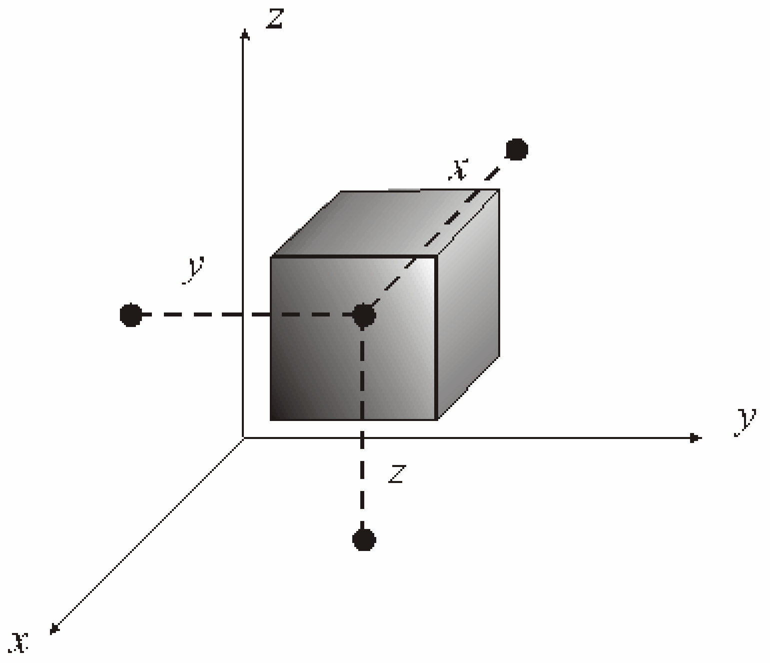
Centro de Masa y Momentos de cuerpos en el espacio

$$m = \iiint_{\Omega} \mu(x, y, z)\, dxdydz$$

$$x_G = \frac{M_{yz}}{m} = \frac{\iiint_{\Omega} x\mu(x, y, z)\, dxdydz}{\iiint_{\Omega} \mu(x, y, z)\, dxdydz}$$

$$y_G = \frac{M_{xz}}{m} = \frac{\iiint_{\Omega} y\mu(x, y, z)\, dxdydz}{\iiint_{\Omega} \mu(x, y, z)\, dxdydz}$$

$$z_G = \frac{M_{xy}}{m} = \frac{\iiint_{\Omega} z\mu(x, y, z)\, dxdydz}{\iiint_{\Omega} \mu(x, y, z)\, dxdydz}$$

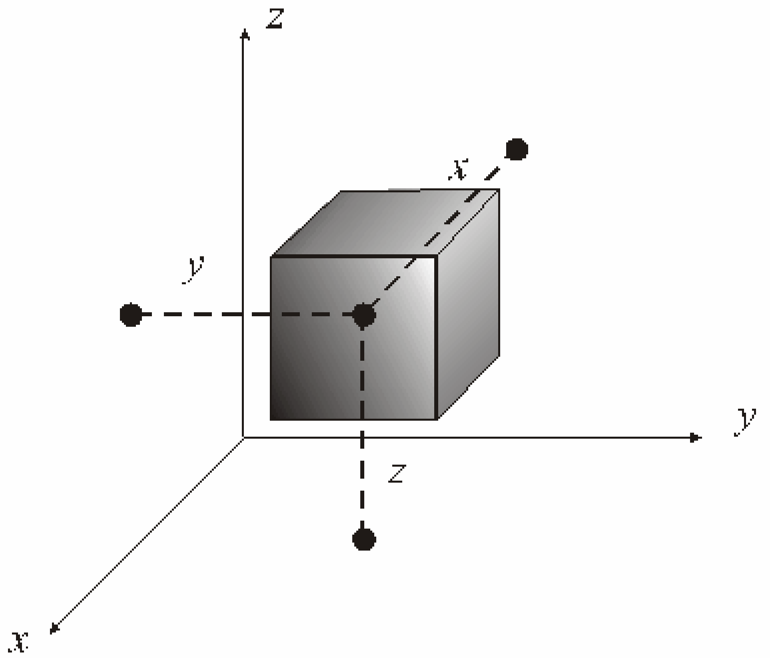


5.3 APLICACIONES
Centro de Masa y Momentos de cuerpos en el espacio

$$I_x = \iiint_{\Omega} \left(y^2 + z^2\right) \mu\left(x, y, z\right) dxdydz$$

$$I_y = \iiint_{\Omega} \left(x^2 + z^2\right) \mu\left(x, y, z\right) dxdydz$$

$$I_z = \iiint_{\Omega} \left(x^2 + y^2\right) \mu\left(x, y, z\right) dxdydz$$



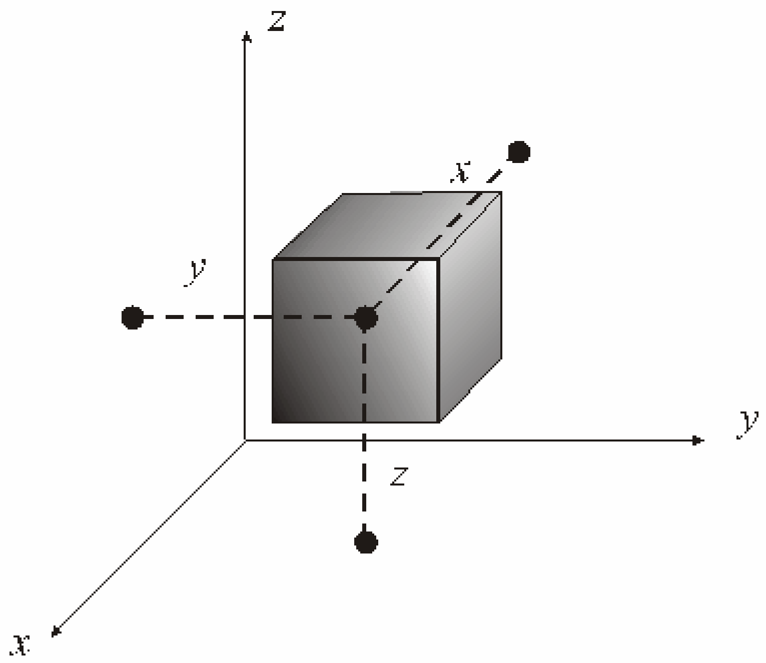
5.3 APLICACIONES

Centro de Masa y Momentos de cuerpos en el espacio

$$I_{xy} = \iiint_{\Omega} z^2 \mu(x, y, z)\, dxdydz$$

$$I_{xz} = \iiint_{\Omega} y^2 \mu(x, y, z)\, dxdydz$$

$$I_{yz} = \iiint_{\Omega} x^2 \mu(x, y, z)\, dxdydz$$



$$I_0 = \iiint_{\Omega} \left(x^2 + y^2 + z^2\right) \mu(x, y, z)\, dxdydz$$