Solving Linear Electronic Circuits by Means of the Fourier Transform

1 Goals

This laboratory session follows a structure very close to the previous one, comprising two independent (but related) sections. First, a (very) basic numerical implementation of the Fourier Transform will be studied so the student can understand some of the peculiarities of the FT when it is calculated by numerical methods, as it is the case of modern computers, oscilloscopes, network analyzers, etc.

The second section pursues to illustrate how the Fourier Transform can be used to solve some (linear) differential equations (or integral equations or integro-differential equations). For this purpose, the analysis of electronic circuits comprising linear components (resistors, capacitors, inductors, etc.) will be considered. Several connected exercises will be proposed to this end. Finally, a short introduction about signal distortion will be considered by analyzing the behavior of conventional oscilloscopes probes.

2 Implementation of the Fourier Transform

The Fourier transform is usually implemented by means of the popular algorithm *Fast Fourier Transform*. Nevertheless, the details of this algorithm are out of the scope of this course. In this section, a basic implementation is considered by means of a workflow similar to the previous session (see Algorithms 1 and 2).

> **!** Since the workflow to calculate the FT is almost identical to the one for the FS, the implementation will not be repeated in this session. Thus, the student can use the functions `fourier_transform.m` and `inverse_fourier_transform.m`, which are available at the course server.

input : Vector `x` containing the values x at which $f(x)$ is evaluated.
input : Vector `f` with the values of the function evaluated at `x`.
input : Vector `xi` with the points ξ at which $F(\xi)$ will be evaluated.
output: Vector `F` with the values of the Fourier transform $F(\xi)$ at the angular frequencies given in `xi`.

Initialize `F` with zeros `%F = zeros(1, length(xi));`
foreach *element* `xi0` *of the vector* `xi` **do**
  `integrand` $\leftarrow f(x)e^{-j\xi_0 x}$;
  `F(xi0)` $\leftarrow$ Value of the integral of `trapz(x,integrand);`
end

Algoritmo 1: Pseudocode for the analysis equation.

input : Vector `xi` with the points ξ at which $F(\xi)$ is evaluated.
input : Vector `F` with the values of the Fourier transform $F(\xi)$ at the angular frequencies given in `xi`.
input : Vector `x` containing the values x at which $f(x)$ will be evaluated.
output: Vector `f` with the values of the function evaluated at `x`.

Initialize `f` with zeros `%f = zeros(1, length(x));`
foreach *element* `x0` *of the vector* `x` **do**
  `integrand` $\leftarrow \frac{1}{2\pi}F(\xi)e^{j\xi x_0}$;
  `f(x0)` $\leftarrow$ Value of the integral of $\frac{1}{2\pi}$`trapz(xi,integrand);`
end

Algoritmo 2: Pseudocode for the synthesis equation.

TAREA

Use the functions `fourier_transform.m` and `inverse_fourier_transform.m`, to perform the following tasks:

1. Numerically calculate the FT of $f(x) = \Pi(x)$ and compare the result with the theoretical one. It is recommended to use a vector x over the range $[-5, 5]$ comprising 2001 points and a vector ξ over the range $[-20, 20]$ with 201 points.

2. Considering the same values as in the previous task, recover $f(x)$ from $F(\xi)$.

3. Numerically calculate the FT of $f(x) = \cos(x)$ and compare the result with the theoretical one. It is recommended to use a vector x over the range $[-5, 5]$ comprising 201 points and a vector ξ over the range $[-20, 20]$ with 2001 points. What are the reasons for the observed differences?

4. In a similar way, calculate the FT of $f(x) = \cos(x)$ and compare the result with the theoretical one. Nevertheless, consider a vector ξ over the range $[-200, 200]$ comprising 2001 points. What new differences can you find?

5. Again, calculate the FT of $f(x) = \cos(x)$ and compare the result with the theoretical one. Nevertheless, consider a vector ξ over the range $[-200, 200]$ comprising 2001 points and a vector x over the range $[-5, 5]$ with 151 points. What are the differences with respect to the previous task?

The strange behavior observed in the last two tasks is due to the sampling of the signal, which results in replicas of the spectrum at positions multiples of $\frac{2\pi}{\Delta x}$. This fact and the impact of truncating the observation domain, which is equivalent to multiplying the signal by a pulse, are the two major facts resulting in differences with respect to the analytical result.

3 Solving electronic circuits fed by arbitrary inputs

Next, the student will be guided through a process which describes a method for solving electronic circuits comprising linear components (resistors, capacitors, inductors, etc.) when arbitrary input signals are applied. This process is mainly theoretical but it requires the use of a computer to solve some sections where the analytical solution is not available in closed-form.

3.1 Kirchhoff's Rules in the spectral domain

TAREA

1. Express the Kirchoff 's rules (Kirchhoff's current law and Kirchhoff's voltage law, see Fig 1) in the spectral domain, i.e., in terms of $I(\omega)$ and $V(\omega)$.

2. Note that Kirchoff's equations for loops have the same format in the time and spectral domains.

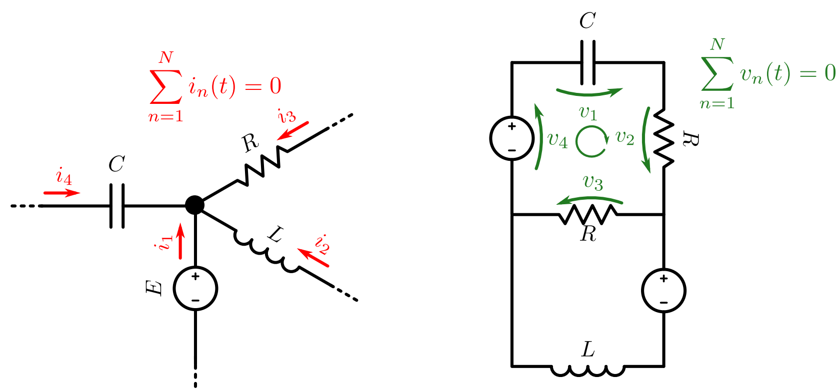


Figure 1: Kirchhoff's Rules.

3.2 Voltage-current equations in the spectral domain

The well-known equations relating voltage and currents for resistors, capacitors and inductors are as follows:

$$i(t) = \frac{v(t)}{R}$$
$$i(t) = C\frac{dv(t)}{dt}$$
$$v(t) = L\frac{di(t)}{dt}$$

TAREA

1. Taking into account the properties of the Fourier transform, express the previous equations in the spectral domain.
2. Note that the ratio $\frac{V(\omega)}{I(\omega)}$ corresponds to the conventional *impedance* of each element.

3.3 Solving linear electronic circuits

Since (a) Kirchhoff's rules are valid in the spectral domain, and (b), in a similar fashion to resistors in DC, the current is proportional to the voltage in the spectral domain, then the electronic circuits can be solved by using "resistors", whose values are : R, $j\omega L$ and $\frac{1}{j\omega C}$.

The previous reasoning enables us to analyze any (linear) circuit in terms of impedances, which are the same ones as studied in previous courses.

Let us consider a basic, but illustrative, analysis of the circuit described in Fig. 2 where $R = 1\,\mathrm{k}\Omega$ and $C = 10\,\mu\mathrm{F}$. Moreover, let us define the input as $v_s(t)$ and the output as $v_c(t)$:

TAREA

1. Determine the transfer function of the system $H(\omega) = \frac{V_c(\omega)}{V_s(\omega)}$. Calculate and represent the amplitude and phase.
2. Do you expect from the system to attenuate low or high frequencies?

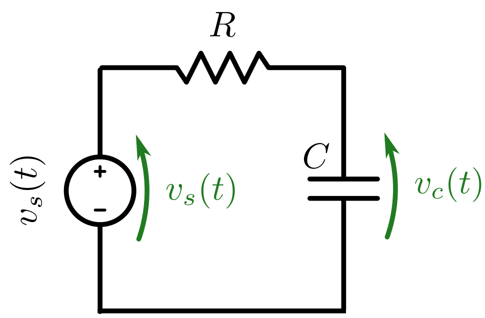


Figure 2: RC circuit.

The previous results are in agreement with the results from previous courses for sinusoidal steady-state responses. Nevertheless, it is important to note that the previous equation actually enable us to solve the circuit for *any* input signal (i.e., not only sinusoidal inputs) as long as we can find the Fourier transform of the input signal. Let's consider a few examples:

TAREA

3. Check that if the input signal is a unit step function $v_s(t) = u(t)$, then the spectrum $V_c(\omega) = H(\omega)V_s(\omega)$ matches the spectrum of the analytical result corresponding to a charging capacitor: $v_c(t) = \left(1 - e^{-\frac{t}{RC}}\right) u(t)$.
Note that this part should be solved *analytically*. For this purpose, determine the analytical spectrum using the closed-form expressions for $H(\omega)$ and $V_s(\omega)$. After that, find (analytically) the FT of $v_c(t)$. Finally, check that both expressions are identical.

4. With the help of a *computer*, plot the voltage $v_c(t)$ for the two following input signals $v_s(t) = \Pi\left(\frac{t}{0.5}\right)$ and $v_s(t) = e^{-10000t^2}$.
In this case, it is recommended to find first the theoretical spectrum (i.e., the closed-form expression) and, after that, compute the inverse FT using numerical techniques (i.e., exploiting the provided numerical functions).

> **!** Regarding the frequency vector, it is recommended to use a vector from $-1\,\text{kHz}$ to $1\,\text{kHz}$ with a step of $1\,\text{Hz}$. Regarding the time vector, it is recommended to use samples from $-0.5\,\text{s}$ to $0.5\,\text{s}$, with a step of $1\,\text{ms}$.

OPCIONAL

5. Repeat the tasks 1, 2 and 4 if the capacitor is replaced by an inductor with $L =$ 30mH.

Even if this technique for solving electronic circuits (and the underlying differential equation) has been illustrated only for a basic circuit, an identical procedure can be followed to solve circuits that are more complex by: 1) obtaining the transfer function using the conventional theory of circuits; 2) calculating by means of analytical or numerical methods the FT of the input signal; 3) calculating the spectrum of the output signal by means of the corresponding product; 4) computing the inverse Fourier Transform.

4 Analysis of an oscilloscope probe

To illustrate how powerful the previous approach is, the analysis of a widespread element, whose details are usually skipped, will be considered: an oscilloscope probe.

The aim of an oscilloscope probe is to transfer the signals to the oscilloscope without distorting (i.e., preserving the shape) the signals. An oscilloscope should ideally behave as an element with a large resistance R_i (usually $1\text{M}\Omega$) in order to produce a minimum perturbation in the circuit under test. This setup is depicted in Fig. 3a.

Nevertheless, there are several difficulties in practice:

- The probe, in order to avoid the impact of noise, consists of a coaxial cable, which introduced a parasitic capacitance C_{coax} in parallel. This capacitance depends on the length of the cable and it is in the order of $100\,\text{pF/m}$.
- In addition, the oscilloscope also has a parasitic capacitance C_i between $10\,\text{pF}$ and $25\,\text{pF}$ in parallel. The internal resistance R_i and the parasitic capacitance C_i are usually given in the corresponding datasheet.

This *nonideal* setup is show in Fig. 3b. Hereinafter, the next values will be considered: $R_i = 1\,\text{M}\Omega$, $C_i = 25\,\text{pF}$ and $C_{coax} = 150\,\text{pF}$. In addition, an additional resistance $R_{coax} \approx 15\,\Omega$ associated to the coaxial will be also considered.

4.1 Analysis of a *1x* oscilloscope probe

In order to understand the impact of the probe in the measured signals, let us analyze the impact when fast changing signals are considered as, for example, pulses $v_i(t) = \Pi\left(\frac{t}{\Delta}\right)$

TAREA

1. Calculate the transfer function $H(\omega) = \frac{V_{osc}(\omega)}{V_i(\omega)}$. Represent the modulus and phase.
2. Represent, with the help of a computer, the signal measured by an oscilloscope $v_{osc}(t)$ if $\Delta = 500\,\text{ms}$.

   ! It is recommended to use a range of frequencies from $-1\,\text{kHz}$ to $1\,\text{kHz}$ with a step of $1\,\text{Hz}$. Regarding the time vector, a reasonable range is from $-0.5\,\text{s}$ to $0.5\,\text{s}$, with a step of $1\,\text{ms}$. Note that $V_i(\omega) = \mathbf{FT}\left\{\Pi\left(\frac{t}{\Delta}\right)\right\} = \Delta\text{sinc}\left(\frac{\Delta}{2}\omega\right)$.

3. Repeat the previous task if $\Delta = 0.5\,\text{ms}$.

   ! It is recommended to used a range of frequencies from $-500\,\text{kHz}$ to $500\,\text{kHz}$ with a step of $50\,\text{Hz}$. Regarding the time vector, a reasonable range is from $-2\,\text{ms}$ and $2\,\text{ms}$, with a step of $10\,\mu\text{s}$.

Note that the signals are almost not distorted, except for the Gibbs phenomenon due to the truncation associated to the numerical simulation.

4.2 Low-frequency analysis of a *10x* oscilloscope probe

The previous probe corresponds to the simplest probe and it does not attenuate the (low-frequency) signals so it is usually referred to as *1x* probe. Despite these probes are very useful for many applications, sometimes the internal resistance of $1\,\mathrm{M}\Omega$ is not enough to avoid perturbing the circuit. Furthermore, this impedance is significantly reduced at high frequencies due to the impact of the parallel capacitor, which behaves as a short-circuit at high frequencies, altering the behavior of the circuit under analysis. For this reason, many probes are equipped with a switch to choose a different working mode known as *10x* (other modes can also be found, working in a similar fashion).

The *10x* probes add two new elements to the standard probe: 1) a resistor $R_{10x} = 9\,\mathrm{M}\Omega$ in series; 2) a *trimmer* (variable) capacitor so that its capacitance C_{10x} can be tuned with the help of a screwdriver. This new setup is shown in Fig. 3c. Note that the coaxial resistance has been removed (R_{coax}) since its impact is negligible after considering the new elements (e.g., the resistor of $9\,\mathrm{M}\Omega$).

Considering a *10x* oscilloscope probe, perform the following tasks:

TAREA

1. Determine the transfer function $H(\omega) = \frac{V_{osc}(\omega)}{V_i(\omega)}$.
2. Let us consider that the new capacitor has a negligible impact at low frequencies ($|Z| = \frac{1}{\omega c_{10x}} \to \infty$) , represent, with the help of a computer, the signal measured by the oscilloscope $v_{osc}(t)$ when $\Delta = 500\,\mathrm{ms}$ (use the same parameters as in that simulation for the *1x* case). What differences do you notice with respect to the 1x case?
3. Repeat the previous task, supposing $\frac{1}{\omega c_{10x}} \to \infty$ again, if $\Delta = 0.5\,\mathrm{ms}$ (use the same parameters as in the previous simulation).

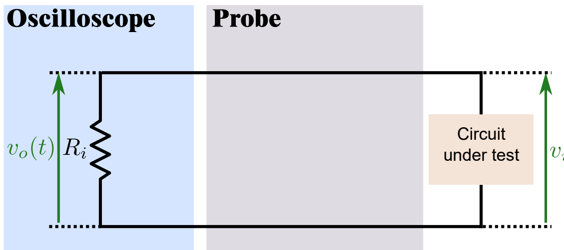
You should have observed that: a) these probes show a resistance of $10\,\mathrm{M}\Omega$ at low frequencies, reducing the perturbation of the circuit under test; b) the probe, even at high frequencies, exhibits an attenuation of x10, which must be compensated by the oscilloscope software; c) if the variable capacitor is not used, the probe strongly alters the shape of the signal even at medium frequencies. In the next section, it will be shown how this variable capacitor can be used to compensate this alteration.

4.3 Signal distortion

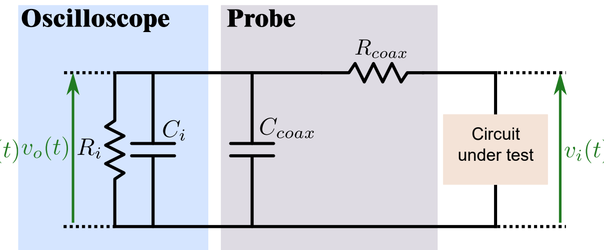
A system is said to be *distortionless* if the output $g(x)$ always preserves the shape of the input $f(x)$:

$$g(t) = Kf(t - t_0)$$

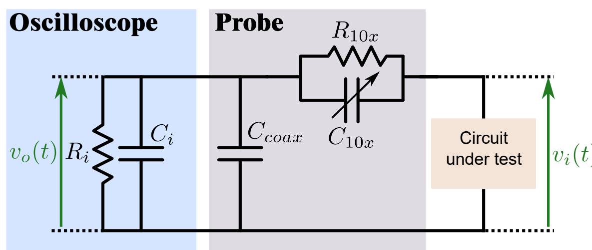
consequently, a *distortionless* system can only delay (or advance) the signal and change the amplitude. Taking into account the previous definition:



(a) Ideal *1x* probe



(b) Nonideal *1x* probe



(c) Nonideal *10x* probe

Figure 3: Ideal and nonideal probes.

TAREA

1. Determine what kind of conditions must meet the transfer function of a distortionless system (i.e., how the system must be in the frequency domain).

2. Taking into account the previous conditions for the transfer function, determine the value of c_{10x} so that the *10x* probe is distortionless.
   Tip: note that the system can be analyzed as a voltage divider (i.e., two impedances in series), in which each impedance is given by a resistor in parallel with one or two capacitors.