Fourier Series in Matlab

1 Goals

This laboratory session is focused on the Fourier series from a theoretical but also practical point of view. First, the student will implement the analysis and synthesis equations with a subsequent validation by comparing the results with some analytical ones. Second, a simplified study of the working principle of frequency multiplier, which is a usual component of radiofrequency devices, is considered.

2 Basic implementation of Fourier Series in Matlab

As it has been described in the lectures, the Fourier Series are defined in terms of the analysis and synthesis equations:

$$a(m) = \frac{1}{X_0} \int_{\langle X_0 \rangle} f_0(x) e^{-jm\xi_0 x} dx \qquad f_0(x) = \sum_{m=-\infty}^{\infty} a(m) e^{jm\xi_0 x}$$

The basic implementation of both equations is sketched in Algorithms 1 and 2. However, several concerns must be taken into account in order to implement these equations. Regarding the analysis equation (Algorithm 1), the vector comprising the integration points `x` must cover an entire fundamental period. In addition, the input signal $f(x)$ is no longer defined as an *anonymous function* as in previous sessions but as a vector comprising the function evaluated at the points described in `x` [1]. Finally, it is important to note that the routine provides the coefficients over a symmetric interval from `-N` to `N`, where `N` is an integer defined by the user.

Regarding the synthesis equation (Algorithm 2), the Fourier coefficients vector is supposed to be defined over a symmetric interval from `-N` to `N`. Hence, the Fourier series is *truncated*. In this case, there are no constraints regarding the observation points `x`.

> **!** In this session, in order to mitigate numerical errors, it is recommended to work with 1001 points per period, which are enough to accurately model the signals considered in these notes. Nevertheless, you must be aware that this value is not optimum and it could not be accurate for many signals.

[1] Actually, this function could be defined as an *anonymous function* as well. However, this implementation is preferred to ease the second part of this session.

input : Vector `x` containing the values x where $f(x)$ is evaluated. The length must be a fundamental period.
input : Vector `f0` with the values of the function evaluated at `x`.
input : Number of the maximum coefficient `N`.
output: Vector `a` with the coefficients from $-N$ to N. Thus, the length of the vector is $2N+1$.

X0 $\leftarrow$ `x(end)-x(1)` `%Fundamental period;`
xi0 $\leftarrow \frac{2\pi}{X_0}$ `%Angular frequency;`
Initialize `a` with zeros `%a = zeros(1, 2*N+1);`
for *m from* $-N$ *to* N **do**
  `%Index ind to access to a so that the coefficient for` $m=-N$ `is at the first position (1) and the coefficient for` $m=N$ `is at the last position` $(2N+1)$
  ind $\leftarrow$ m + N + 1 ;
  `integrand` $\leftarrow \frac{1}{X_0} f_0(x) e^{-jm\xi_0 x}$;
  `a(ind)` $\leftarrow$ Value of the integral of `trapz(x,integrand)`;
end

Algorithm 1: Pseudocode for the analysis equation.

input : Vector `a` with the coefficients of the Fourier series. The coefficients are expected to be in a symmetric range from $-N$ to N
input : `X0` fundamental period of the signal.
input : Vector `x` containing the values x where $f_0(x)$ will be evaluated. The length must be a fundamental period.
output: Vector `f` with the values of $f_0(x)$ evaluated at `x`

N $\leftarrow$ `(length(a)-1)/2` `%Max. coeff.;`
xi0 $\leftarrow \frac{2\pi}{X_0}$ `%Angular frequency;`
Initialize `f0` with zeros `%f0 = zeros(1, length(x));`
for *m from* $-N$ *to* N **do**
  f0 $\leftarrow$ f0 + $a(m)e^{jm\xi_0 x}$;
end

Algorithm 2: Pseudocode for the synthesis equation.

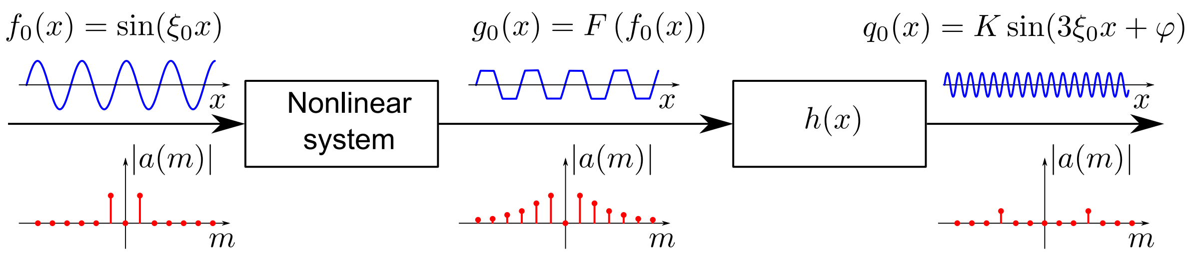

Figure 1: Working principle of a frequency multiplier.

TASK

1. Implement the analysis and synthesis equations following the pseudocode in Algorithm 1 and 2..
2. Use those routines to numerically compute the FS of $f_0(x) = \Pi_{0,1}(x) = \sum_{n=-\infty}^{\infty} \Pi(x-3n)$. Compare the result with the theoretical one.
3. Retrieve the signal $f_0(x)$ from the previous coefficients. What differences do you observe? What are the reasons for these differences? Experiment with $N = 5$ and $N = 25$.
4. Compute the FS of $f_0(x)$ defined over a fundamental period X_0 as $f_0(x) = e^{-x/X_0}$, $x \in [0, X_0)$ wherein $X_0 = 2$. Compare the result with the theoretical one.

3 Analysis of a system by means of Fourier series

Generating very high frequency sinusoidal signals (e.g., hundreds of GHz) is a challenging task. In this section, the foundations for generating a signal at a given frequency from another signal at a lower frequency are introduced from the point of view of Fourier series.

3.1 Working principle

An approach to generate high frequency sinusoidal signals is to use two cascaded systems as depicted in Fig. 1. The first system is a nonlinear and memoryless system (e.g., a diode). Thus, the signal is no longer a pure sinusoid but a general periodic signal. Since the signal is still periodic (with the same fundamental period), it can be represented in terms of a FS (i.e., a sum of sines and cosines), whose coefficients will be generally nonzero.

The second system is a linear and shift-invariant system, which works as a filter with coefficients $H(n) = H(n\xi_0) = 0$ except for the target frequency. Thus, the output signal only comprises terms at the desired frequency.

3.2 Analysis of the system

Next, several tasks are proposed in order to illustrate the behavior of these devices. For this purpose, let us suppose that the relationship between input and output signals of the nonlinear system is defined as:

$$g(x) = \begin{cases} A & f(x) > A \\ -A & f(x) < -A \\ f(x) & \text{otherwise} \end{cases},$$

this system can be easily implemented by means of the following function:

```
function g = non_linear_system(f, A)
g = f; %Initial assignment.

ind = f>A; %Indexes at which f is greater than A
g(ind)= A;

ind = f<-A; %Indexes at which f is less than -A
g(ind) = -A;
```

Note that if A is set to zero, then the output signal is zero. Conversely, if the parameter A is too large (larger than the amplitude of the input signal), then the intermediate output signal $g_0(x)$ will remain unmodified and, therefore, the final output signal $q_0(x)$ will be zero again. Thus, there will be at least one optimum working point (i.e., a point at which the output power is maximum) for some intermediate value of A.

Let $f_0(x) = C\sin(\xi_0 x)$, $\xi_0 = 2\pi$, $C = 1$, and a vector x defined over $x \in [-\frac{X_0}{2}, \frac{X_0}{2})$ with 1001 points, perform the following tasks:

TASK

1. Plot the signals $f_0(x)$ and $g_0(x)$ for different values of A (e.g., 0, $0.25C$, $0.75C$, C and $1.25C$).

2. Numerically compute the average power of $f_0(x)$ and $g_0(x)$ for different values of A.

3. Compute and compare the Fourier coefficients corresponding to the FS $f_0(x)$ and $g_0(x)$ for different values of A.

4. Numerically compute the average power of $f_0(x)$ and $g_0(x)$ from the coefficients of the FS (it is recommended that you choose $N \geq 15$).

5. Let us consider that the desired frequency of the final signal $q_0(x)$ is $3\xi_0$. Plot the ratio of the input signal, $f_0(x)$, power to the output signal, $q_0(x)$, power, which is usually known as *power efficiency*, over the interval $A \in [0, C]$. For this purpose, numerically compute the power efficiency for several values of A using Matlab. What is the optimum working point?

6. Let $A = 0.5$, plot the input and output signals, $f_0(x)$ and $q_0(x)$. In addition to the difference in frequency, could you tell any other relevant difference?