Chapter 2: Worksheet

If you have any doubts about the notation of the different sources, you can check the conversion sheet available in the virtual campus.

Problem 1

Show that the norm induced by the inner product meets the requirements to be qualified as a norm.

Problem 2

Show that the metric induced by the norm product meets the requirements to be qualified as a metric.

Problem 3

Determine the norm induced by the inner product $\|f(x)\| = (\langle f(x), f(x)\rangle)^{\frac{1}{2}}$ for the following fictions and sort these functions according to its "size" (i.e., its norm):

- $f_1(x) = e^{-x^2}$, note that $\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$.
- $f_2(x) = e^{-|x|}$
- $f_3(x) = \frac{1}{x^2+1}$, note that, by integrating by parts ($x = \tan u$ and $dx = \sec^2 u\, du$), you can show that $\int \frac{1}{(x^2+a^2)^2}dx = \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right)$
- $f_4(x) = \frac{x}{x^2+1}$, similar to the previous case $\int \frac{x^2}{(x^2+a^2)^2}dx = \frac{-x}{2(x^2+a^2)} + \frac{1}{2a}\tan^{-1}\left(\frac{x}{a}\right)$
- $f_5(x) = e^{-|x|}\cos(\xi_0 x)$, the following integral can be useful $\int e^{ax}\cos^n(bx)\,dx = \frac{e^{ax}\cos^{n-1}(bx)}{a^2+n^2b^2}\left[a\cos(bx) + nb\sin(bx)\right] + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\cos^{n-2}(bx)\,dx$
- $f_6(x) = e^{-|x|}\sin(\xi_0 x)$, the following integral can be useful $\int e^{ax}\sin^n(bx)\,dx = \frac{e^{ax}\sin^{n-1}(bx)}{a^2+n^2b^2}\left[a\sin(bx) - nb\cos(bx)\right] + \frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{ax}\sin^{n-2}(bx)\,dx$

Problem 4

Calculate the metric induced by the norm between $f_1(x) = e^{-x^2}$ and $f_2(x) = e^{-|x|}$, i. e., $d(f_1(x), f_2(x))$. The following results from statistics can be useful: $\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$. In addition to the previous expression, by completing the squares it is easy to show that $\int_0^{\infty} e^{-(ax^2+bx+c)}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{(b^2-4ac)/4a}\text{erfc}\left(\frac{b}{2\sqrt{a}}\right)$ wherein erfc is the complementary error function, which you can evaluate using a calculator.

Problem 5

Calculate the norm induced by the inner product $\|x(n)\| = (\langle x(n), x(n)\rangle)^{\frac{1}{2}}$ of the following functions and sort theses functions according to its size (i.e., its norm):

- $x_1(n) = e^{-n^2}$
- $x_2(n) = e^{-|n|}$

In order to solve these problem, the formula for the geometric series sum, which is available in the tables of the course, can be helpful.

Problem 6

Problems 4.1, 4.2 and 4.3 from the book by Emilio Gago (these exercises have been partially solved in the problem-solving session).

Problem 7

Determine the power and energy for each of the following signals::

- $f_1(x) = e^{7t}u(t)$
- $f_2(x) = e^{j\left(2t+\frac{\pi}{4}\right)}$
- $f_3(x) = \cos(x)$
- $x_1(n) = \frac{1}{2^n}u(n)$
- $x_2(n) = e^{j\left(\frac{\pi}{2}n+\frac{\pi}{8}\right)}$
- $x_3(n) = \cos(\frac{\pi}{4}n)$

Note: the unit step functions $u(x)$ y $u(n)$ are defined as $u(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$ and $u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$.

Problem 8

Determine whether or not each of the following signals is periodic:

- $f_1(x) = 2e^{j\left(x+\frac{\pi}{4}\right)}u(x)$
- $x_2(n) = u(n) + u(-n)$
- $x_3(n) = \sum_{k=-\infty}^{\infty} \{\delta(n-4k) - \delta(n-1-4k)\}$

Note: the discrete-variable impulse function is defined as $\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$

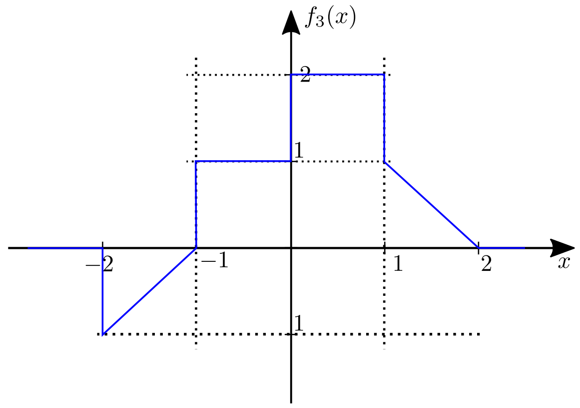
Problem 9

Determine the fundamental period of the signal $f(x) = 2\cos(10x+1) - \sin(4x-1)$

Problem 10

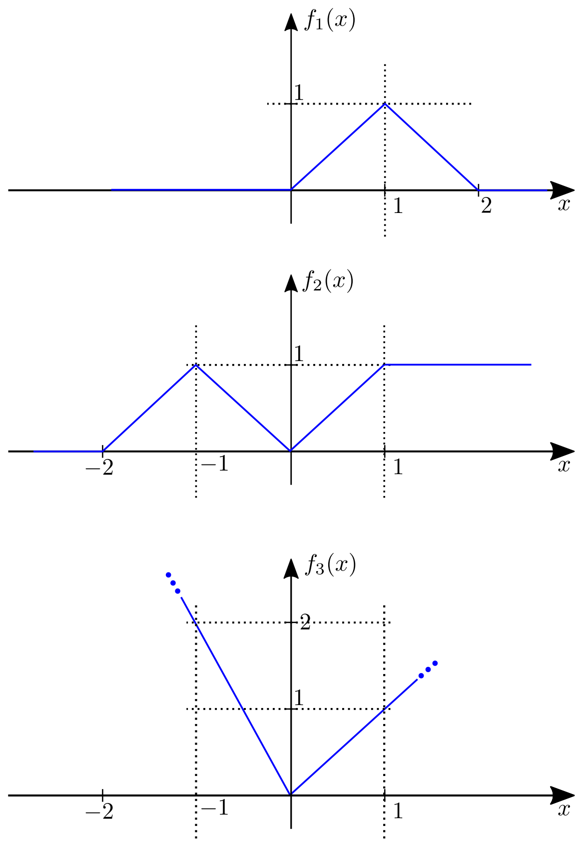
A continuous-variable signal $f(x)$ is shown below. Sketch and label carefully each of the following signals:

1. $f(x-1)$
2. $f(2-x)$
3. $f(2x+1)$
4. $f(4-\frac{x}{2})$
5. $[f(x)+f(-x)]\,u(x)$



Problem 11

Determine and sketch the even and odd parts of the signals depicted in the figure below. Label your sketches carefully.



Problem 12

Determine whether or not each of the following signals is periodic. If the signal is periodic, determine its fundamental period:

1. $f(x) = 3\cos\left(4x + \frac{\pi}{3}\right)$
2. $f(x) = e^{j(\pi x - 1)}$
3. $f(x) = \cos^2\left(2x - \frac{\pi}{3}\right)$
4. $f(x) = \mathcal{E}v\left\{\cos\left(4\pi x u(t)\right)\right\}$, wherein $\mathcal{E}v\left\{\cdot\right\}$ denotes even part of the signal.
5. $f(x) = \mathcal{E}v\left\{\sin\left(4\pi x u(t)\right)\right\}$, wherein $\mathcal{E}v\left\{\cdot\right\}$ denotes even part of the signal.
6. $f(x) = \sum_{n=-\infty}^{\infty} e^{2x-n}$.

Problem 13

Let $f(x)$ be a continuous-variable signal, and let

$$g_1(x) = x(2t) \text{ and } g_2(x) = f(x/2)$$

The signal $g_1(x)$ represents a speeded up version of $f(x)$ in t he sense that the duration of the signal is cut in half. Similarly, $g_2(x)$ represents a slowed down version of $f(x)$ in the sense that the duration of the signal is doubled. Consider the following statements:.

- If $f(x)$ is periodic, then $g_1(x)$ is periodic.
- If $g_1(x)$ is periodic, then $f(x)$ is periodic.
- If $f(x)$ is periodic, then$g_2(x)$ is periodic.
- If $g_2(x)$ is periodic, then $f(x)$ is periodic.

For each of the statements, determine whether it is true, and if so, determine the relationship between the fundamental periods of the two signals considered in the statement. I f the statement is not true, produce a counterexample to it.

Problem 14

In this problem, we explore several of the properties of even and odd signals.

- Show that if if $x(n)$ is and odd signal, then:
$$\sum_{n=-\infty}^{\infty} x(n) = 0$$
- Show that if $x_1(n)$ is an odd signal and $x_2(n)$ is an even signal, then $x_1(n)x_2(n)$ is an odd signal.
- Let $x(n)$ be an arbitrary signal, with even and odd parts denoted by $x_e(n)$ and $x_o(n)$. Show that:
$$\sum_{n=-\infty}^{\infty} x^2(n) = \sum_{n=-\infty}^{\infty} x_e^2(n) + \sum_{n=-\infty}^{\infty} x_o^2(n)$$
- Although the previous parts have been stated in terms of discrete-variable signals, the analogous properties are also valid in continuous time. To demonstrate this, show that:
$$\int_{-\infty}^{\infty} f^2(x)dx = \int_{-\infty}^{\infty} f_e^2(x)dx + \int_{-\infty}^{\infty} f_o^2(x)dx$$
where $f_e(x)$ and $f_o(x)$ are the even and odd parts of $f(x)$, respectively.