Impulse response of linear and shift-invariant systems

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1 Exercise 2: Properties of the impulse response

1.1 Question a)

In order to calculate the collection of impulse responses, we set the impulse as an impulse at an arbitrary position $f(x) = \delta(x - x')$ and, taking into account the definition of the system:

$$h(x; x') = \int_x^{\infty} \delta(y - x') e^{x-ay} dy = \begin{cases} 0 & x > x' \\ e^{x-ax'} & x < x' \end{cases} = u(x' - x) e^{x-ax'}$$

1.2 Question b)

First approach: using the definition of the system with $f(x) = u(x)$:

$$g(x) = \int_x^{\infty} f(y) e^{x-ay} dy = \int_x^{\infty} u(y) e^{x-ay} dy = \begin{cases} \int_0^{\infty} e^{x-ay} dy & x < 0 \\ \int_x^{\infty} e^{x-ay} dy & x > 0 \end{cases}$$
$$= \begin{cases} e^x \frac{-1}{a} e^{-ay}\Big|_0^{\infty} = \frac{e^x}{a} & x < 0 \\ e^x \frac{-1}{a} e^{-ay}\Big|_x^{\infty} = \frac{e^{(1-a)x}}{a} = & x > 0 \end{cases} = \frac{e^x}{a} u(-x) + \frac{e^{(1-a)x}}{a} u(x)$$

Second approach: using the collection of impulse responses

$$g(x) = \int_{-\infty}^{\infty} f(x') h(x; x') dx' = \int_{-\infty}^{\infty} u(x') u(x' - x) e^{x-ax'} dx' = \int_x^{\infty} u(x') e^{x-ax'} dx'$$

kindly note that the previous integral is equivalent to the one from the first approach so it can be solved following the same steps.

1.3 Question c)

If $a = 1$, the system would be shift-invariant since $h(x; x')$ depends on $x - x'$. In particular, $h(x; x') = u(x' - x) e^{x-x'} = h(x - x')$ and, consequently, the impulse response can be found by setting $x' = 0$, i.e., $h(x) = u(-x) e^x$

1.4 Question d)

We can analyze the properties by inspecting the impulse response $h(x) = u(-x)e^x$.

The system would be memoryless if and only if the impulse response is Dirac delta, which is not the case. Hence, the system *has* memory.

The system would be causal if and only if $h(x) = 0, \quad \forall x < 0$, which not the case. Hence, the system is *not* causal.

Finally, the system is stable if and only if $\int_{-\infty}^{\infty} |h(x)|\, dx < \infty$:

$$\int_{-\infty}^{\infty} |h(x)|\, dx = \int_{-\infty}^{\infty} e^x u(-x) dx = \int_{-\infty}^{0} e^x dx = 1$$

Thus, the system is *stable*.

1.5 Question e)

In order to find the inverse system, we must find a system whose impulse response $h^{-1}(x)$ meets the following condition:

$$h(x) * h^{-1}(x) = \delta(x) \tag{1}$$

In order to find that impulse response, we consider first the tip given by the exercise:

$$h(x) \quad * \quad \delta'(x) \qquad = \qquad \frac{dh(x)}{dx} \qquad = \qquad \frac{d}{dx}(u(-x)e^x)$$

It is important to note that $h(x) = e^x u(-x)$ is *not* a continuous function and, therefore, we must be careful when calculating the first derivative:

$$\begin{aligned} h(x)*\delta'(x) = \frac{d}{dx}(u(-x)e^x) = \frac{d}{dx}(u(-x))e^x + u(-x))\frac{d}{dx}e^x = \frac{d}{dx}(u(-x))e^x + u(-x))e^x \\ = -u'(-x) + h(x) = -\delta(-x) + h(x) = -\delta(x) + h(x) \quad (2) \end{aligned}$$

In the previous steps, we have considered that the first derivative of $u(-x)$ is $-\delta(x)$ because the gap is negative.

Next, we will try to express (2) as in (1):

$$h(x) * [-\delta'(x)] = \delta(x) - h(x)$$

Taking into account that $h(x) = h(x) * \delta(x)$ and applying the distributive property:

$$h(x) * [\delta(x) - \delta'(x)] = \delta(x)$$

Hence, the inverse system is characterized by the following impulse response $h^{-1}(x) = \delta(x) - \delta'(x)$.