Problem solving session: Convolution

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1 Exercise 3

Let us start by working on the left hand side:

$$[f(x) * h(x)]*g(x) = c(x)*g(x) = \int_{\infty}^{\infty} c(y)g(x-y)dy = \int_{\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x')h(y-x')dx'\right] g(x-y)dy$$

Next, both integrals will be merged (kindly note that this step assumes that he Fubini-Tonelli theorem requirements are met) and finally the change of variable $y' = y - x'$ is done:

$$[f(x) * h(x)]*g(x) = \int_{\infty}^{\infty} \int_{-\infty}^{\infty} f(x')h(y-x')g(x-y)dx'dy = \int_{\infty}^{\infty} \int_{-\infty}^{\infty} f(x')h(y')g(x-x'-y')dx'dy'$$

Next, similar operations are accomplished on the right hand side:

$$f(x) * [h(x) * g(x)] = f(x) * c_1(x) = \int_{\infty}^{\infty} f(x')c_1(x-x')dx'$$

where

$$c_1(x) = \int_{\infty}^{\infty} h(y)g(x-y)dy \rightarrow c_1(x-x') = \int_{\infty}^{\infty} h(y)g(x-x'-y)dy$$

therefore:

$$f(x) * [h(x) * g(x)] = \int_{\infty}^{\infty} f(x') \left[\int_{\infty}^{\infty} h(y)g(x-x'-y)dy\right] dx'$$

next, we merge again both integrals assuming that the Fubini-Tonelli requirements are met so an equivalent result is achieved:

$$f(x) * [h(x) * g(x)] = \int_{\infty}^{\infty} \int_{\infty}^{\infty} f(x')h(y)g(x-x'-y)dx'dy$$

2 Exercise 4

2.1 First section

It will be useful to express $h_2(n)$ as follows:

$$h_2(n) = \frac{3}{2}u(n) - \frac{1}{2}\delta(n)$$

wherein

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$u(n) = \begin{cases} 1 & n \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

and it is also useful to consider the result of the following convolutions:

$$x(n) * u(n) = \sum_{m=-\infty}^{n} x(m)$$

$$x(n) * \delta(n) = x(n)$$

First, the convolution $w(n) = u(n) * h_1(n)$ is derived taking into account the previous properties

$$w(n) = u(n) * h_1(n) = \sum_{m=0}^{n} \left(-\frac{1}{2}\right)^m$$

The previous summation corresponds to a geometrical series so the result is $S = \frac{1-r^{n+1}}{1-r}$ if $n > 0$. Otherwise, there are no terms in the summation and the result would be 0. If we merge both cases (n>0 and n<0) by means of the unit step function, we can express the result as:

$$w(n) = \frac{2}{3}\left(1 - \left(-\frac{1}{2}\right)^{n+1}\right)u(n)$$

Finally, the overall convolution is:

$$y(n) = w(n) * h_2(n) = \sum_{m=-\infty}^{n} \frac{3}{2}w(m) - \frac{1}{2}w(n)$$

$$y(n) = \sum_{m=0}^{n}\left(1 - \left(-\frac{1}{2}\right)^{m+1}\right) - \frac{1}{2}w(n) = (n+1)u(n) - \sum_{m=0}^{n}\left(-\frac{1}{2}\right)^{m+1} - \frac{1}{2}w(n)$$

$$= (n+1)u(n) + \frac{1}{2}\sum_{m=0}^{n}\left(-\frac{1}{2}\right)^{m} - \frac{1}{2}w(n) = (n+1)u(n)$$

2.2 Second section

We will express the second signal as in the previous section:

$h_2(n) = \frac{3}{2}u(n) - \frac{1}{2}\delta(n)$

Next, we calculate $w(n)$

$w(n) = h_1(n) * h_2(n) = \frac{3}{2}\sum_{k=0}^{n}\left(-\frac{1}{2}\right)^k - \frac{1}{2}h_1(n) = \left(1 - \left(-\frac{1}{2}\right)^{n+1}\right)u(n) - \frac{1}{2}\left(-\frac{1}{2}\right)^n u(n) = u(n)$

The final convolution is:

$$y(n) = x(n) * w(n) = u(n) * u(n) = \sum_{m=-\infty}^{n} u(m) = \sum_{m=0}^{n} u(m) = (n+1)u(n)$$

and, therefore, the result is the same as in the previous section

2.3 Third section

First, we find $x(n) * h_2(n)$

$x(n) * h_2(n) = a^n u(n) - aa^{n-1}u(n-1) = a^n(u(n) - u(n-1) = a^n\delta(n) = \delta(n)$

and, consequently

$x(n) * [h_1(n) * h_2(n)] = [x(n) * h_2(n)] * h_1(n) = \delta(n) * h_1(n) = h_1(n) = \sin(8n)$