Problem-Solving Session

Other inner product definitions

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Exercise 1: Non-orthogonal bases

(based on the problem 3.66 from the book by Oppenheim)
The best approximation theorem is extremely important in that it show that each coefficient α_m is independent of all the other α_m. Thus, if we add more terms to the approximation, those coefficients will not change.
In contrast to this, let us consider the polynomial Taylor Series. For example, in the case of an exponential function:

$$f(x) = e^x = 1 + x + \frac{x^2}{2} + \dots$$

As it will be shown, if this series is finite, then the best coefficients are different from the previous ones. In particular, let us consider just three basis functions: $e_0(x) = 1$, $e_1(x) = x$ y $e_2(x) = x^2$.

a) Are the $e_m(x)$ orthogonal over the interval $0 \leq x \leq 1$?

b) Consider an approximation of $f(x)$ over the previous interval by means of a single basis function $\hat{f}(x) = \alpha_0 e_0(x)$. Find the value of the coefficient α_0 that minimizes the energy in the error signal $E = \left\| f(x) - \hat{f}(x) \right\|$ over the interval .

c) Consider the same exercise but taking into account two basis functions so $\hat{f}(x) = \alpha_0 e_0(x) + \alpha_1 e_1(x)$. For this purpose, solve the simultaneous equations:

$$\frac{\partial E}{\partial \alpha_0} = 0 \quad \textit{and} \quad \frac{\partial E}{\partial \alpha_1} = 0$$

Exercise 2: Other definitions of the inner product

Consider the following mapping between signal spaces:

$$
\begin{aligned}
& S(-1,1)\times S(-1,1)\to\mathbb{C}\\
& \qquad (f(x),g(x))\mapsto\langle f(x),g(x)\rangle_w=\int_{-1}^{1}f(x)g^*(x)w(x)dx,
\end{aligned}
$$

where $S(-1,1)$ represents the signal space of real independent variable in the interval $-1\leq x\leq 1$, and $w(x)=\sqrt{1-x^2}$ is a weight function.

a) Determine whether or not the previous mapping is an inner product for $S([-1,1])$.

b) Determine whether the *Chebyshev polynomials of second kind*, defined as follows:

$$
\begin{aligned}
U_0(x) &= 1\\
U_1(x) &= 2x\\
U_{n+1}(x) &= 2xU_n(x)-U_{n-1}(x)
\end{aligned}
$$

and satisfying the following property:

$$U_n(\cos\theta)=\frac{\sin((n+1)\theta)}{\sin\theta}$$

are orthogonal according to the previous definition.

Tip: the next change of variable is advised $x=\cos\theta$.