The Z-Transform

Appendix B

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Outline

Introduction

- The Z-transform is considered as a discrete-time equivalent of the Laplace transform.
- Thus, it can be understood as an analytic continuation of the discrete-variable Fourier transform.

$$e^{j\Omega} \rightarrow z = |z|\, e^{j\Omega} = z' + jz''$$

- Again, this transform does not follow the General Theory of Transforms and, therefore, it cannot be represented by means of inner products and linear combinations.
- It enables us to calculate a transform (and, consequently, to operate in the transformed domain) even in cases in which the Fourier transform does not converge.
  - E.g.: an unstable system cannot be usually represented by means of the FT but it can be represented in terms of the Z-transform in many cases.
- The Fourier Transform can be found from the Z-transform in many cases.

Definition of the Z-Transform

- The Z-transform of a sequence $x(n)$ is defined as:

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}}.$$

- A representation in terms of modulus and phase enables us to find a clear relation with the FT:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \left|z\right|^{-n} e^{-j\Omega n} = \mathbf{FT}\left(x(n) \left|z\right|^{-n}\right).$$

- The previous relation suggests that the convergence will be improved in many cases thanks to the term $\left|z\right|^{-n}$.

Definition and ROC

- The convergence of the previous summation is not guaranteed at all the points of the complex plane.
  - The TZ convergences if any of the convergence criteria for the FT is met:

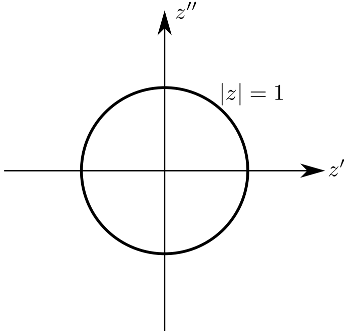
$$\sum_{n=-\infty}^{\infty} |x(n)|\,|z|^{-n} < \infty \text{ or } \sum_{n=-\infty}^{\infty} |x(n)|^2\,|z|^{-2n} < \infty.$$

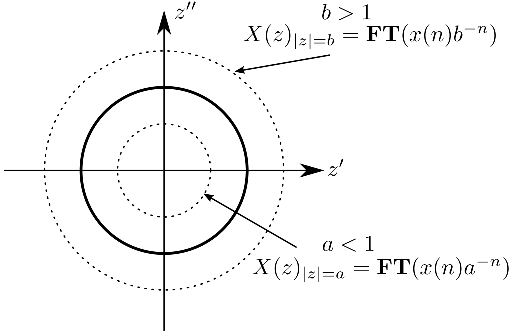
- The Region of Convergence (ROC) is the complex plane region such that:

$$\text{ROC} = \left\{z = z' + jz'' \text{ such that } |X(z)| < \infty\right\}.$$

- The relation with the FT suggests the ROC will depend on the modulus $|z|$ and, in fact, the ROC consists of a ring in the z-plane centered about the origin.
- The FT can be retrieved from the Z-transform if the unit circle is included the ROC.

Definition and ROC





Region of Convergence for Z-transforms of Unilateral sequences

- In order to demonstrate the previous fact about the shape of the ROCs, let us consider the following two cases.

Lemma

If the Z-transform of a causal sequence ($x(n) = 0$ for all $n < 0$) (conditionally or absolutely) converges at a point z_0, then the Z-transform of the sequence absolutely converges for all z_1 such that $|z_1| > |z_0|$.

Lemma

If the Z-transform of a anticausal sequence($x(n) = 0$ for all $n \geq 0$)(conditionally or absolutely) converges at a point z_0, then the Z-transform of the sequence absolutely converges for all z_1 such that $|z_1| < |z_0|$.

- Taking into account the two previous lemmas and the fact that each sequence can be decomposed into a causal and anticausal sequence, then the ROC, which is the intersection of each individual ROC, will be a ring.

Convergence of causal sequences - Proof

Proof.

Let $x(n)$ be a causal sequence such that $X(z_0) = \sum_{n=0}^{\infty} x(n) z_0^{-n} < \infty$.
If the series convergences, then the sequence $x(n)z_0^{-n}$ is bounded:

$$\left| x(n) z_0^{-n} \right| = |x(n)| \, |z_0|^{-n} \leq M, \; M < \infty$$

Hence:

$$\left| x(n) z_1^{-n} \right| = \left| x(n) z_0^{-n} \right| \left| \frac{z_1}{z_0} \right|^{-n} \leq M\rho^{-n}, \rho = \frac{|z_1|}{|z_0|}$$

Since $\rho^{-1} < 1$, the series $\sum_{n=0}^{\infty} M\rho^{-n}$ converges and, therefore, the series $\sum_{n=0}^{\infty} \left| x(n) z_1^{-n} \right|$ also converges. Thus, we conclude that $X(z_1)$ is absolutely convergent. □

Exercise

Show the equivalent lemma for anticausal sequences.

Regions of Convergence - I

- Next, we explore a number a properties about the region of convergence.
  - If the signal is of finite length, whose support is over an interval $[N_1, N_2]$, then the ROC is the entire z-plane, except possibly $z = 0$ and/or $|z| \to \infty$.
    - $|z| \to \infty \in$ ROC if and only if $N_1 \geq 0$.
    - $z = 0 \in$ ROC if and only if $N_2 \leq 0$.
  - If the signal is right-sided and the Z-transform converges at z_0,then the Z-transform absolutely converges at all points z_1, $|z_1| > |z_0|$, except for possibly $|z| \to \infty$.
    - If the support is the interval $[N_1, \infty)$, then $|z| \to \infty \in$ ROC if and only if $N_1 \geq 0$.
  - If the signal is left-sided and the Z-transform converges at z_0, then the Z-transform absolutely converges at all points z_1, $|z_1| < |z_0|$, except for possibly $z = 0$.
    - If the support is the interval $(-\infty, N_2]$, then $z = 0 \in$ ROC if and only if $N_2 \leq 0$.
  - If the signal is two-sided and the Z-transform converges at a point z_0, then the ROC is a ring containing the point z_0.

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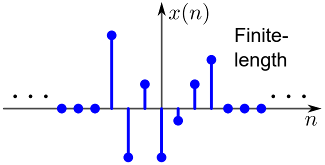
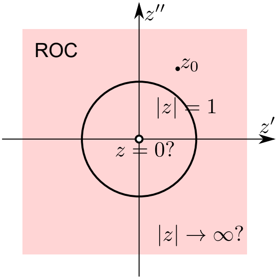
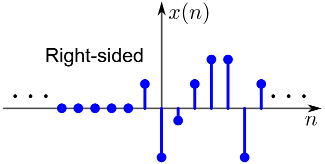
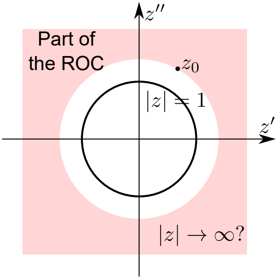
Regions of Convergence - II

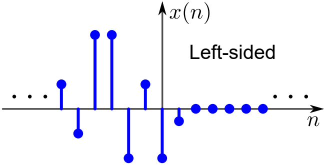


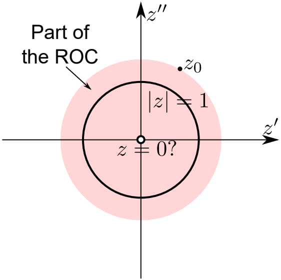


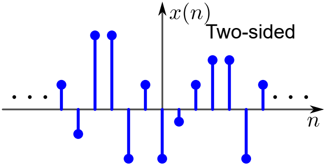


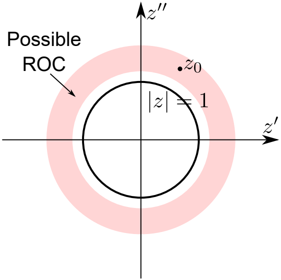


Regions of Convergence - III









The inverse Z-transform

- As previously discussed, the Z-transform can be calculated at each point $z = |z|e^{j\Omega}$ in the ROC as:

$$X(z) = \mathbf{FT}\left\{x(n)|z|^{-n}\right\}.$$

- Taking into account the inverse Fourier Transform:

$$x(n)|z|^{-n} = \frac{1}{2\pi}\int_{2\pi} X(z)e^{j\Omega n}d\Omega,$$

and, therefore

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(z)|z|^{n}e^{j\Omega n}d\Omega.$$

- The previous expression is usually expressed by means of a complex-variable integral by means of the following change of variable $z = |z|e^{j\Omega}$, $\frac{dz}{d\Omega} = j|z|e^{j\Omega} = jz$

$$\boxed{x(n) = \frac{1}{2\pi j}\oint_{|z|} X(z)z^{n-1}dz}.$$

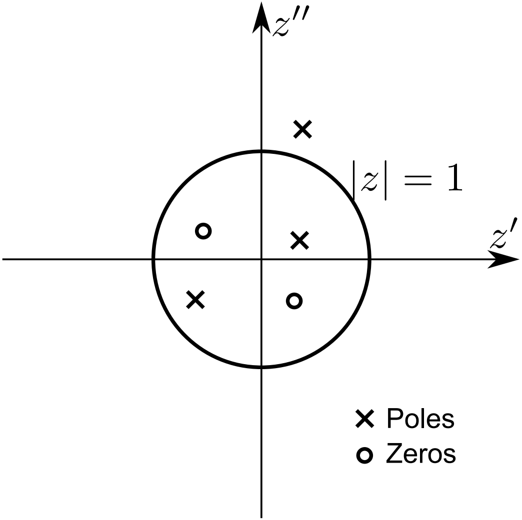
Rational Z-Transforms

- It is very usual to use rational Z-transform, i.e., transforms described by the quotient of two polynomials in z (or z^{-1}):

$$X(z) = \frac{\beta_M z^M + \beta_{M-1} z^{M-1} + \ldots + \beta_0}{\alpha_N z^N + \alpha_{N-1} z^{N-1} + \ldots + \alpha_0}$$

- It is important to note that the previous factors can be repeated several times.
  - The number of repetitions of any factor is known as multiplicity (or order).
- Again, these transform can be characterized by means of a pole-zero plot.
- Rational Z-transforms will arise in the following cases:
  - Systems described by linear constant coefficient difference equations
  - Signals described by exponentials

Pole-zero plots



ROC for rational Z-Transforms

ROC - rational TZ

If the Z-transform $X(z)$ of $x(n)$ is rational, then the ROC is bounded by poles or extends to infinity.

ROC - right-sided $x(n)$

If the Z-transform $X(z)$ of $x(n)$ is rational, and $x(n)$ is right-sided, then the ROC is the region of the z-plane outside the outermost pole. Moreover, if $x(n)$ is causal, then the ROC also includes $|z| \to \infty$.

ROC - left-sided $x(n)$

If the Z-transform $X(z)$ of $x(n)$ is rational, and $x(n)$ is left-sided, then the ROC is the region of the z-plane inside the innermost nonzero pole, except for possibly $z = 0$. Moreover, if $x(n)$ is anticausal, then the ROC also includes $z = 0$.

ROC $x(n)$ - two-sided

If the Z-transform $X(z)$ of $x(n)$ is rational, and $x(n)$ is two-sided, then the ROC is a ring between two adjacent poles. Unless additional information is available, there could be more than one choice.

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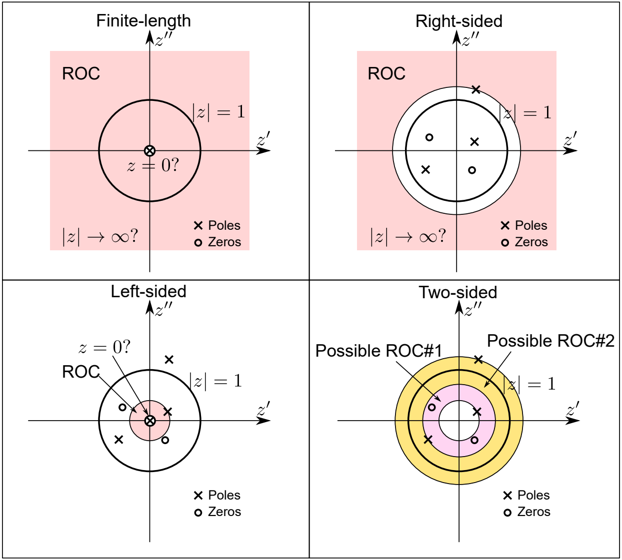
Rational Z-transform - ROC



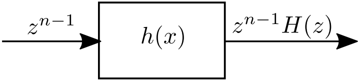
Properties of the Z-transform

Linearity	$\alpha x(n)+\beta y(n)$	$\alpha X(z)+\beta Y(z)$	At least $R_X \cap R_Y$
Shifting	$x(n-n_o)$	$X(z)z^{-n_0}$	R_X (origin?)
Mult. by $e^{j\Omega_0 n}$	$x(n)e^{j\Omega_0 n}$	$X(ze^{-j\Omega_0})$	R_X
Mult. by z_0^n	$x(n)z_0^n$	$X(\frac{z}{z_0})$	$\|z_0\|R_X$
Mult. by a^n	$x(n)a^n$	$X(\frac{z}{a})$	$\|a\|R_X$
Complex conjugate	$x^*(n)$	$X^*(z^*)$	R_X
Reversal	$x(-n)$	$X(1/z)$	$1/R_X$
Scaling	$x(n/p)$	$X(z^p)$	$[R_X]^{\frac{1}{p}}$
Convolution	$x(n)*y(n)$	$X(z)Y(z)$	At least $R_X \cap R_Y$
First difference	$x(n)-x(n-1)$	$\left(1-z^{-1}\right)X(z)$	At leastR_X
Accumulation	$\sum_{p=-\infty}^{n} x(p)$	$\frac{1}{1-z^{-1}}X(z)$	At least$R_X \cap \|z\| > 1$
Multiplication by n	$nx(n)$	$-\frac{dX(z)}{dz}$	R_X

Z-Transform pairs

Impulse	$\delta(n)$	1	$\forall z$
Shifted impulse	$\delta(n-n_o)$	z^{-n_o}	$\forall z$
Heaviside	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\|>1$
	$-u(-n-1)$	$\frac{1}{1-z^{-1}}$	$\|z\|<1$
Exponentials	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\|>\|a\|$
	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|>\|a\|$
	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\|<\|a\|$
	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|<\|a\|$
Right-sided cosine	$\cos(\Omega_0 n)u(n)$	$\frac{1-z^{-1}\cos\Omega_0}{z^{-2}-2z^{-1}\cos\Omega_0+1}$	$\|z\|>\|1\|$
Right-sided sine	$\sin(\Omega_0 n)u(n)$	$\frac{z^{-1}\sin\Omega_0}{z^{-2}-2z^{-1}\cos\Omega_0+1}$	$\|z\|>\|1\|$
Prod. cosine and right-sided exp.	$a^n\cos(\Omega_0 n)u(n)$	$\frac{1-z^{-1}a\cos\Omega_0}{a^2z^{-2}-2az^{-1}\cos\Omega_0+1}$	$\|z\|>\|a\|$
Prod. sine and right-sided exp.	$a^n\sin(\Omega_0 n)u(n)$	$\frac{z^{-1}a\sin\Omega_0}{a^2z^{-2}-2z^{-1}a\cos\Omega_0+1}$	$\|z\|>\|a\|$

Characterization of LSI systems using the Z-transform



- The signals $x(n) = z^{n-1}$ are also eigenfunctions of the LSI systems:

$$z^{n-1} * h(n) = \sum_{k=-\infty}^{\infty} z^{n-k-1} h(k) = z^{n-1} \sum_{k=-\infty}^{\infty} z^{-k} h(k) = z^{n-1} H(z)$$

- Hence, the response to an input $x(n)$ is:

$$y(n) = x(n) * h(n) = \frac{1}{2\pi j} \oint_{|z|} X(z) \left(z^{n-1} * h(n) \right) dz = \frac{1}{2\pi j} \oint_{|z|} X(z) H(z) z^{n-1} dz$$

- Consequently, $H(z)$ plays the role of transfer function by relating input and output as:

$$\boxed{Y(z) = X(z)H(z)}$$

Stability analysis of LSI systems using the Z-transform

- It is worthwhile to remember that a LSI system is stable if and only if the impulse response is absolutely summable.
- According to the previous condition, it can be stated that the FT exists for each impulse response of each stable system.
- This fact together with the absolute convergence in all the ROC except for possibly at the boundary results in:

Stable LSI system

A LSI is stable if and only the ROC includes the unit circle $|z| = 1$ and it is not an empty ring.

Causality analysis of LSI systems using the Z-transform

- On the other hand, a *causal LSI system* has a transfer function $H(z)$ whose ROC is the exterior of a circle, *including* $|z| \to \infty$.

Causal LSI system with rational transfer function

A LSI system with rational transfer function $H(z)$ is causal if and only if: (a) the ROC is the exterior of a circle outside the outermost pole; (b) the order of the numerator cannot be greater than the order of the denominator with $H(z)$ expressed in terms of polynomials in z.

- For example, $H(z) = \frac{z^3 - 2z^2 + z}{z^2 + \frac{1}{4}z + \frac{1}{8}}$ cannot be causal, without even knowing his ROC, because the numerator is of higher order than the denominator.
- If $H(z)$ is rational and the system is causal and stable, then all the poles must be inside the unit circle.

Systems defined by a difference equation

Exercise

Analyze the stability of the following system, which is known to be causal, defined by the following linear constant-coefficient difference equation:

$$y(n)+\frac{1}{4}y(n-1)-\frac{1}{8}y(n-2)=x(n).$$

Solution

$$H(z)=\frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}=\frac{1}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

The system is stable

Systems represented by block diagrams

Example

Find the transfer function of the following systems.