The Laplace Transform

Appendix A

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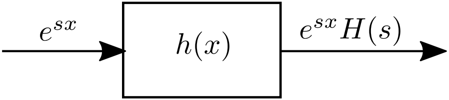
Outline

Introduction

- The Laplace Transform can be understood as an analytic continuation of the Fourier Transform (FT).
$$j\xi \rightarrow s = \sigma + j\xi$$
- The basis functions are given by:
$$\varphi(x;s) = e^{sx},\ s \in \mathbb{C}$$
however the conventional signal space algebra cannot be used since these functions are complex-variable functions habitual.
  - Thus, this transformation cannot be modeled as the previous ones by means of the General Theory of Transforms.
- It enable us to define a transform (and, therefore, capabilities to operate in the transformed domain) even if the FT does not exist.
  - E.g,: an unstable system will not have, in general, FT, in terms of conventional functions. However, its Laplace transform does exist in many cases.
- Many FT can be found from the Laplace Transform.
  - However, there are exceptions, as for example when the FT requires to be represented by means of distributions.

Basis functions and linear systems



- The basis functions $\varphi(x;s) = e^{sx}$ are also eigenfunctions of the LSI systems:

$$\varphi(x;s) * h(x) = \int_{-\infty}^{\infty} e^{s(x-x')} h(x') dx'$$
$$= e^{sx} \int_{-\infty}^{\infty} e^{-sx'} h(x') dx' = \varphi(x;s) H(s)$$

wherein the following term has been defined:

$$H(s) = \int_{-\infty}^{\infty} h(x') e^{-sx'} dx'$$

this transformation is referred to as the Laplace Transform

Definition and Region of Convergence for Laplace Transforms

- The Laplace Transform of a signal $f(x)$ is defined as:

$$\boxed{F(s) = \int_{-\infty}^{\infty} f(x) e^{-sx} dx}$$

- The previous transform may not convergence at the entire complex plane.
- The Region of Convergence (ROC) is defined as the region of the complex plane such that:

$$\text{ROC} = \left\{ s = \sigma + j\xi \text{ such that } \int_{-\infty}^{\infty} \left| f(x) e^{-\sigma x} \right| dx < \infty \right\}$$

  - Convergence can exist (conditionally convergent) even if the previous condition is not met.

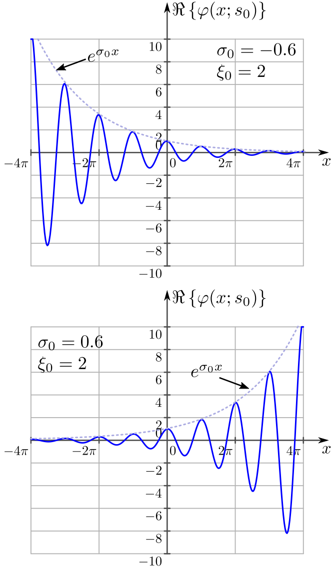
Analysis of the basis functions - I

- The basis functions can be represented as:

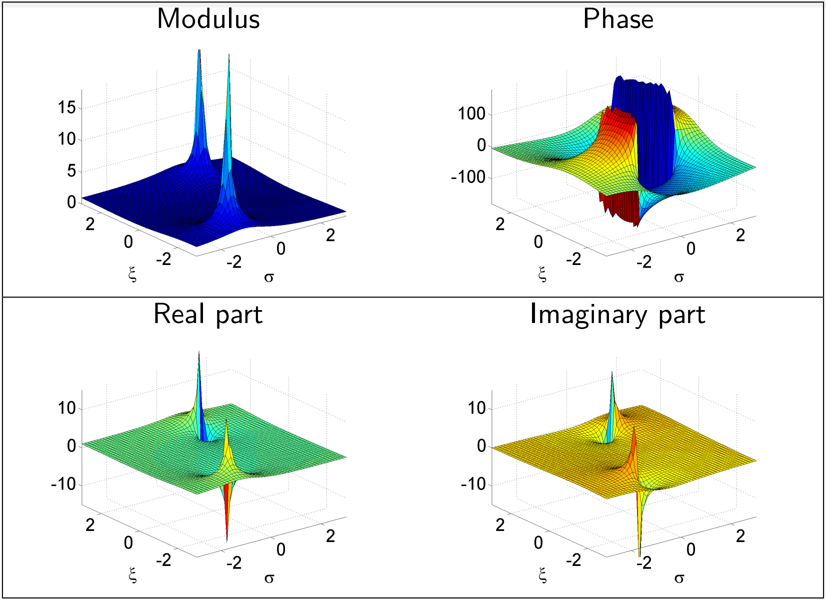
$$\varphi(x;s) = e^{\sigma} e^{j\xi} = e^{\sigma}\left(\cos\left(j\xi\right) + j\sin\left(j\xi\right)\right)$$

- It is relevant to note the following properties:
  - They are not periodic except for $\sigma = 0$
  - They are not square integrable.
  - If $\Re(s) = \sigma > 0$, then $\lim_{x\rightarrow\infty} \varphi(x;s) \rightarrow \infty$
  - If $\Re(s) = \sigma > 0$, then $\lim_{x\rightarrow-\infty} \varphi(x;s) \rightarrow \infty$

Analysis of the basis functions - II



Three-dimensional visual representation



Relation with the Fourier Transform

- If the straight line $\sigma = 0$ is in the ROC, then the Fourier Transform can be calculated from the LT.
  - This condition is sufficient to guarantee the existence of the FT but it is not necessary.
- Moreover, the LT can be calculated from the FT as:

$$\mathbf{TL}\left(f(x)\right) = F(s) = \int_{-\infty}^{\infty} f(x)e^{-sx}dx = \int_{-\infty}^{\infty} f(x)e^{-\sigma x}e^{-j\xi x}dx$$
$$= \int_{-\infty}^{\infty} g(x,\sigma)e^{-j\xi x}dx = \mathbf{FT}\left(g(x,\sigma)\right)$$

- Hence:

$$\boxed{F\left(s = \sigma + j\xi\right) = \mathbf{TL}\left(f(x)\right) = \mathbf{FT}\left(f(x)e^{-\sigma x}\right)}$$

- Thus, the LT will convergence $f(x)e^{-\sigma x}$ is square integrable or absolutely integrable.
  - Hereafter, the two remaining Dirichlet conditions are supposed to be satisfied.

Regions of Convergence

- The absolutely integrability criterion for $f(x)e^{-\sigma x}$ will be considered here as it is the most restrictive one.
  - If the signal is of finite-length, and absolutely integrable, then the ROC is the entire s-plane.
  - If the signal is of infinite-length and right-sided and the point $s_0 = \sigma_0 + j\xi_0$ is in the ROC, then all values of $s = \sigma_1 + j\xi$ for which $\sigma_1 \geq \sigma_0$ will also be in the ROC.
  - If the signal is of infinite-length and left-sided and the point $s_0 = \sigma_0 + j\xi_0$ is in the ROC, then all values of $s = \sigma_1 + j\xi$ for which $\sigma_1 \leq \sigma_0$ will also be in the ROC.
  - If the signal is two-sided and $s_0 = \sigma_0 + j\xi_0$ is in the ROC, then the ROC will consist of a strip in the s-plane that includes the point s_0.

Region of Convergence - Example

Exercise

Calculate the Laplace Transform of $f_1(x) = e^{-ax}u(x)$ and $f_2(x) = -e^{-ax}u(-x)$ specifying the corresponding ROCs

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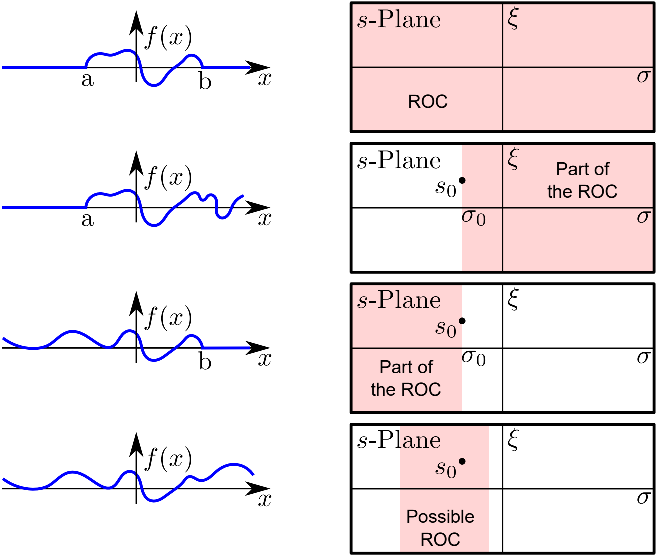
Solution

$F_1(s) = \frac{1}{s+a}$ with $\Re\{s\} > -\Re\{a\}$ and $F_2(s) = \frac{1}{s+a}$ with $\Re\{s\} < -\Re\{a\}$

Regions of convergence



The inverse Laplace transform

- As previously discussed, the LT can be calculated at each line $s = \sigma_0 + j\xi$ in the ROC:

$$F(s = \sigma_o + j\xi) = \mathbf{FT}\left\{f(x)e^{-\sigma_o x}\right\}.$$

- Taking into account the inverse Fourier Transform:

$$f(x)e^{-\sigma_o x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma_o + j\xi)e^{j\xi x}d\xi,$$

and, therefore

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma_o + j\xi)e^{(\sigma_0 + j\xi)x}d\xi.$$

- The previous expression is usually expressed by means of a complex-variable integral by means of the following change of variable $s = \sigma_o + j\xi$

$$\boxed{f(x) = \frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s)e^{sx}ds}.$$

Rational Laplace Transforms

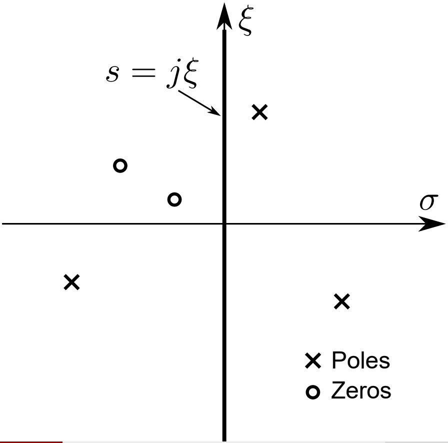
- It is very usual to use rational Laplace Transforms, i.e., transforms described by the quotient of two polynomials:

$$F(s) = \frac{\beta_M s^M + \beta_{M-1} s^{M-1} + \ldots + \beta_0}{\alpha_N s^N + \alpha_{N-1} s^{N-1} + \ldots + \alpha_0} = K \frac{\prod_{k=1}^{M} (s - s_{z_k})}{\prod_{k=1}^{N} (s - s_{p_k})}$$

- It is important to note that the previous factors can be repeated several times.
  - The number of repetitions of any factor is known as multiplicity (or order).
- Roots of the numerator are referred to as zeros.
- Roots of the denominator are referred to as poles.
- Rational Laplace transforms will arise in the following cases:
  - Systems described by differential equations
  - Exponential signals.

Pole-zero plots

- Laplace Transforms are usually represented by means of pole-zero plots.
  - If the Laplace transform is rational, then this plot together withe multiplicity completely defines the signal.



ROC for rational Laplace Transforms

ROC for rational Laplace Transforms

If the Laplace transform is rational, then the ROC is bounded by poles or extends to infinity. Moreover, no poles are contained in the ROC.

Right-sided signal with rational TL

If the Laplace Transform is rational and the signal is right-sided, then the ROC is the region in the s-plane to the right of the rightmost pole.

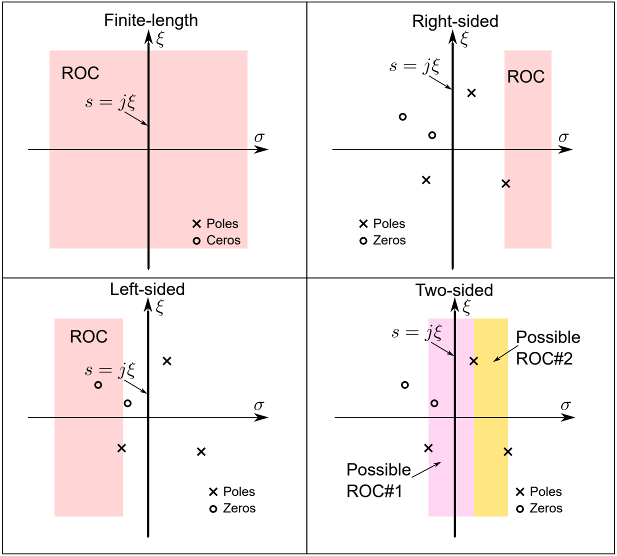
Left-sided signal with rational TL

If the Laplace Transform is rational and the signal is left-sided, then the ROC is the region in the s-plane to the right of the leftmost pole.

Two-sided signal with rational TL

If the Laplace Transform is rational and the signal is two-sided, then the ROC is a vertical strip in the s-plane bounded by two adjacent poles.

ROC for Rational Laplace Transforms



Properties of the Laplace Transform

Linearity	$\alpha f(x)+\beta g(x)$	$\alpha F(s)+\beta G(s)$	At least $R_F \cap R_G$
Shifting	$f(x-x_o)$	$F(s)e^{-sx_o}$	R_F
Mult. by a complex exp.	$f(x)e^{s_o x}$	$F(s-s_o)$	Shifted R_F
Complex conjugate	$f^*(x)$	$F^*(s^*)$	R_F
Reversal	$f(-x)$	$F(-s)$	Reversed R_F
Scaling	$f(ax)$	$\frac{1}{\|a\|}F(\frac{s}{a})$	Scaled R_F
Convolution	$f(x)*g(x)$	$F(s)G(s)$	At least $R_F \cap R_G$
First derivative	$\frac{df(x)}{dx}$	$sF(s)$	At least R_F
High-order derivatives	$\frac{d^n f(x)}{dx^n}$	$s^n F(s)$	At least R_F
Integration	$\int_{-\infty}^{x} f(y)dy$	$\frac{1}{s}F(s)$	At least $R_F \cap \Re\{s\}>0$
Multiplication by x	$-xf(x)$	$\frac{dF(s)}{ds}$	R_F

Highlighted properties differ from the equivalent ones in the case of the unilateral Laplace Transform (see appendix).

Laplace transform pairs

Impulse	$\delta(x)$	1	$\forall s$
Shifted impulse	$\delta(x-x_o)$	e^{-sx_o}	$\forall s$
Impulse derivatives	$\delta^{(n)}(x)$	s^n	$\forall s$
Heaviside	$\pm u(\pm x)$	$\frac{1}{s}$	$\Re\{s\} \gtrless 0$
	$xu(x)$	$\frac{1}{s^2}$	$\Re\{s\} > 0$
	$\pm\frac{x^{n-1}}{(n-1)!}u(\pm x)$	$\frac{1}{s^n}$	$\Re\{s\} \gtrless 0$
Exponentials, $\alpha \in \mathbb{R}$	$\pm e^{-\alpha x}u(\pm x)$	$\frac{1}{s+\alpha}$	$\Re\{s\} \gtrless -\alpha$
	$\pm\frac{x^{n-1}}{(n-1)!}e^{-\alpha x}u(\pm x)$	$\frac{1}{(s+\alpha)^n}$	$\Re\{s\} \gtrless -\alpha$
Right-sided cosine	$\cos(\xi_0 x)\,u(x)$	$\frac{s}{s^2+\xi_0^2}$	$\Re\{s\} > 0$
Right-sided sine	$\sin(\xi_0 x)\,u(x)$	$\frac{\xi_0}{s^2+\xi_0^2}$	$\Re\{s\} > 0$
Prod. cosine and right-sided exp.	$e^{-\alpha x}\cos(\xi_0 x)\,u(x)$	$\frac{s+\alpha}{(s+\alpha)^2+\xi_0^2}$	$\Re\{s\} > -\alpha$
Prod. sine and right-sided exp.	$e^{-\alpha x}\sin(\xi_0 x)\,u(x)$	$\frac{\xi_0}{(s+\alpha)^2+\xi_0^2}$	$\Re\{s\} > -\alpha$

Characterization of LSI systems using the LT - Stability

- It is worthwhile to remember that a LSI system is stable if and only if the impulse response is absolutely integrable.
- According to the previous condition, and assuming that the two remaining Dirichlet conditions are also met, it can be stated that the FT exists for each impulse response of each stable system.

Stable LSI system

A LSI system is stable if and only if the ROC included the imaginary axis $s = j\xi$

Characterization of LSI systems using the LT - Causality

- On the other hand, the ROC associated with the transfer function $H(s)$ for a *causal LSI system* is a right-half plane because $h(x)$ is right-sided.
    - This condition is necessary but it is, in general, *not* sufficient.
    - Nonetheless, it is possible to show that if $H(s)$ is rational, then this condition is necessary and sufficient.

Causal LSI system with rational H(s)

If the transfer function for a LSI system $H(s)$ is rational, then the system is causal if and only if the ROC is the right-half plane to the right of the rightmost pole.

Region of Convergence - Example

Exercise

If the signal $f(x)$ is the input to a system with impulse response $h(x) = -e^{-ax}u(-x) + e^{-ax}u(x)$, is it possible to calculate the output by means of the Laplace Transform?

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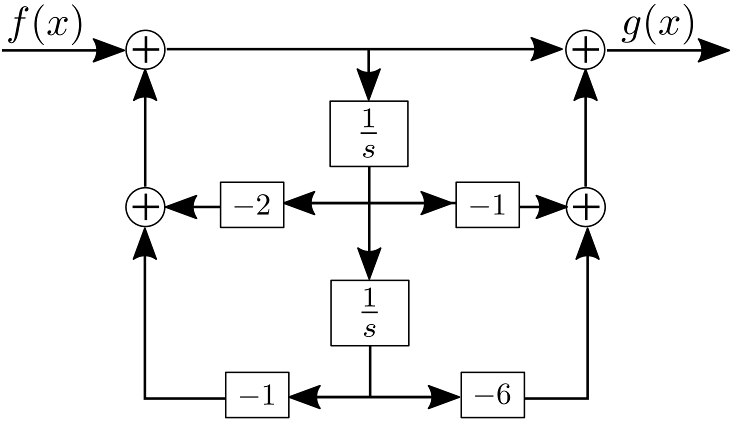
Solution

Yes... and no.

It is not possible to calculate the transfer function $H(s)$ because the ROCs do not overlap.

However, it is possible to decompose the system in terms of causal and anticausal responses $h(x) = h_1(x) + h_2(x)$ and calculate the output as $g(x) = f(x) * h_1(x) + f(x) * h_2(x)$ by independently solving each convolution.

Transfer function for block diagram representations



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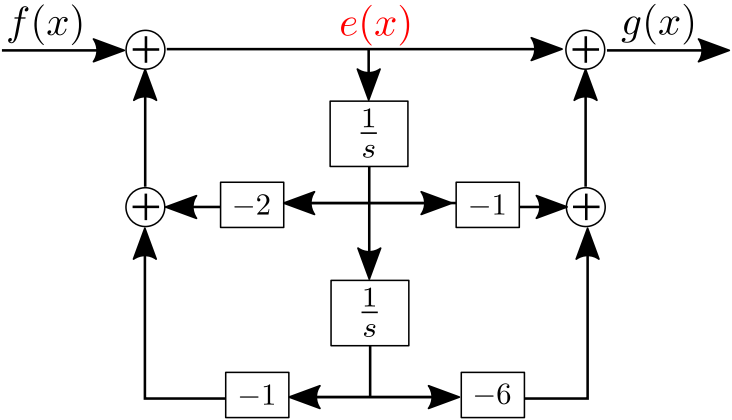

Example - Differential equations

Exercise

Consider a causal LSI system for which the input $f(x)$ and output $g(x)$ satisfy the linear constant-coefficient differential equation
$\frac{d^2f(x)}{dx^2} - \frac{df(x)}{dx} - 6f(x) = \frac{d^2g(x)}{dx^2} - g(x)$. Determine $H(s)$. Is the system stable? Calculate the impulse response.

Exercise

Consider a system with transfer function $H(s) = \frac{s^2-s-6}{s^2-1}$, which is known to be stable. Find the impulse response $h(x)$.

Example - Differential equations

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Solution

The transfer function can be decomposed as $H(s) = \frac{s^2-s-6}{s^2-1} = \frac{(s-3)(s+2)}{(s+1)(s-1)}$. There are three possible ROCs bounded by poles but the only one corresponding to a stable system is the strip $-1 < \Re\{s\} < 1$. After both polynomials to reduce the order of the numerator and by decomposing in terms of partial-fraction expansion $H(s) = \frac{(s-3)(s+2)}{\left(s^2-1\right)} = 1 - \frac{5+s}{(s-1)(s+1)} = 1 - \frac{3}{s-1} + \frac{2}{s+1}$ and, therefore,

Example of polynomial long division

$$
\begin{array}{r|l}
s^2 - s - 6 & s^2 - 1 \\ \cline{2-2}
-\quad s^2 \qquad - 1 & \qquad 1 \\ \cline{1-1}
-s - 5 &
\end{array}
$$

$$\left(s^2-1\right)(+1)+(-s-5)=s^2-s-6$$

$$
\begin{array}{r|l}
6s^3 - 2s^2 + \ \ s + 3 & s^2 - \ \ s + 1 \\ \cline{2-2}
-\quad 6s^3 - 6s^2 + 6s \qquad & \qquad 6s + 4 \\ \cline{1-1}
4s^2 - 5s + 3 & \\
-\quad 4s^2 - 4s + 4 & \\ \cline{1-1}
-s - 1 &
\end{array}
$$

$$\left(s^2-s+1\right)(6s+4)+(-s-1)=6s^3-2s^2+s+3$$

Appendix:
The unilateral Laplace Transform

The unilateral Laplace Transform

- As illustrated, the Laplace transform is often applied to right-sided signals.
- Thus, it is usual to choose a different definition, known as unilateral Laplace Transform or, depending on the author, simply Laplace Transform:

$$\boxed{F(s) = \int_0^{\infty} f(x) e^{-sx} dx}$$

- Most of properties and transform pairs (for right-sided signals) remain the same with two prominent exceptions regarding derivatives:

$$f'(x) \leftrightarrow sF(s) - f(0)$$
$$f''(x) \leftrightarrow s^2 F(s) - sf(0) - f'(0)$$

Appendix:

Initial- and final-value theorems

Initial- and final-value theorems

- Under the specific constraints that $f(x) = 0 \, \forall x < 0$ and that $f(x)$ contains no impulses or high-order singularities at the origin, then the initial- and final-value theorems state that:

$$f(0^+) = \lim_{s \to \infty} sF(s)$$
$$\lim_{x \to \infty} f(x) = \lim_{s \to 0} sF(s)$$

- These theorems are very useful because they enable us to easily calculate important values of the function without calculating the inverse transform (e.g, by means of partial-fraction expansion).

Appendix:

Partial-fraction expansion

Partial-fraction expansion

- If the Laplace Transform is rational, then there is a systematic way to calculate the inverse transform.

Step#1 Normalization by dividing both the numerator and denominator by α_M

$$H(s) = \frac{\beta_N s^M + \beta_1 s + \ldots + \beta_0}{\alpha_M s^N + \ldots + \alpha_1 s + \alpha_0} \Rightarrow H(s) = \frac{u_N s^M + \ldots + + u_1 s + u_0}{a_M s^N + \ldots + a_1 s + 1}$$

Step#2 Reduction of the numerator order if $M \geq N$

$$H(s) = \frac{u_N s^M + \ldots + + u_1 s + u_0}{a_M s^N + \ldots + a_1 s + 1} = c_{M-N} s^{M-N} + c_{M-N-1} s^{M-N-1} + \ldots + c_1 s + c_0$$
$$+ \frac{b_{N-1} s^{N-1} + \ldots + + b_1 s + b_0}{a_M s^N + \ldots + a_1 s + 1}$$

note that $a_M = 1$.

Partial-fraction expansion

the coefficients b_k and c_k can be found by equating the coefficients from both sides:

$$u_N s^M + \ldots + + u_1 s + u_0 = \left(c_{M-N} s^{M-N} + c_{M-N-1} s^{M-N-1} + \ldots + c_1 s^1\right)$$
$$\left(a_M s^N + \ldots + a_1 s + 1\right) + b_{N-1} s^{N-1} + \ldots + + b_1 s + b_0$$

It is also possible to perform this step by means of a polynomial division.

Partial-fraction expansion

Step#3 Expand the remaining fraction in terms of simpler rational fraction:

$$H_{red}(s) = \frac{b_{N-1}s^{N-1} + \dots + + b_1 s + b_0}{a_M s^N + \dots + a_1 s + 1} = \frac{A_{11}}{s-\rho_1} + \frac{A_{12}}{(s-\rho_1)^2} + \dots + \frac{A_{1l_1}}{(s-\rho_1)^{l_1}} +$$

$$+ \frac{A_{21}}{s-\rho_2} + \frac{A_{22}}{(s-\rho_2)^2} + \dots + \frac{A_{2l_2}}{(s-\rho_2)^{l_2}} +$$

$$+ \dots + \frac{A_{R1}}{s-\rho_R} + \frac{A_{R2}}{(s-\rho_R)^2} + \dots + \frac{A_{Rl_R}}{(s-\rho_R)^{l_R}} = \sum_{i=1}^{R} \sum_{k=1}^{l_i} \frac{A_{ik}}{(s-\rho_i)^{l_i}}$$

where ρ_r are the R distinct roots of the denominator and l_k is the multiplicity of the k-th root of the denominator.

This expansion can be carried out as long as the order of the numerator is less than the order of the denominator.

The coefficients can be calculated as:

$$\boxed{A_{ik} = \frac{1}{(l_i - k)!} \frac{d^{l_i-k}}{ds^{l_i-k}} \left[(s-\rho_i)^{l_i} H_{red}(s) \right] \bigg|_{s=\rho_i}}$$

Partial-fraction expansion - Example

Let us consider an example for which the system is characterized by the following differential equation:

$$\frac{d^2g(x)}{dx^2} + 4\frac{dg(x)}{dx} + 3g(x) = \frac{df(x)}{dx} + 2f(x),$$

whose transfer function is given by:

$$H(s) = \frac{G(s)}{F(s)} = \frac{s+2}{s^2+4s+3}.$$

Thus, it can be factorized as:

$$H(s) = \frac{s+2}{(s+1)(s+3)}$$

Let us consider that the system is stable, find the ROC and the impulse response.

Partial-fraction expansion - Example

If the input signal is $f(x) = e^{-x}u(x)$, the LT of the output is given by:

$$G(s) = \frac{s+2}{(s+1)^2(s+3)}$$

The corresponding partial-fraction decomposition is:

$$\frac{s+2}{(s+1)^2(s+3)} = \frac{A_{11}}{(s+1)} + \frac{A_{12}}{(s+1)^2} + \frac{A_{21}}{(s+3)}$$

and the coefficients are given by:

$$A_{11} = \frac{d}{ds} G(s)(s+1)^2 \Big|_{s=-1} = \frac{1}{4} \quad A_{12} = G(s)(s+1)^2 \Big|_{s=-1} = \frac{1}{2} \quad A_{21} = G(s)(s+3)|_{s}$$

and, consequently:

$$g(x) = \left[\frac{1}{4}e^{-x} + \frac{1}{2}xe^{-x} - \frac{1}{4}e^{-3x}\right]u(x)$$