

# Representation of a continuous-variable signal by a discrete-variable signal: Sampling

## Chapter 11

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# Outline

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  - Relation between normalized and real frequencies
- 3 Modeling a CV LSI system by means of a DV LSI system
- 4 Examples



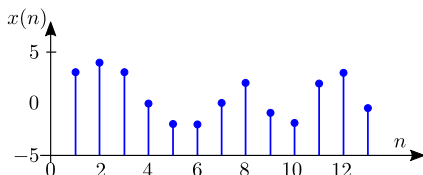
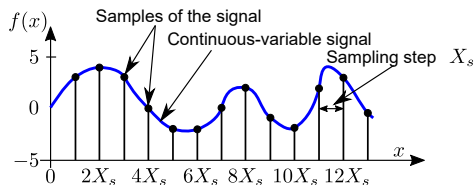
# Introduction

- In this chapter, it will be shown how a **continuous-variable signal** can be represented by its **samples** if some conditions are met.
- The previous fact enables us to perform the **digital processing** of continuous-variable signals.
  - Implementation of continuous-variable systems by means of discrete-variable systems.
  - Analysis of continuous-variable signals by means of digital processing.
- In a practical context, sampled signals are also quantized because they are represented by a **finite number of bits**.
  - **Quantization** approaches and their impact in the signal will be studied in later courses.



# Introduction

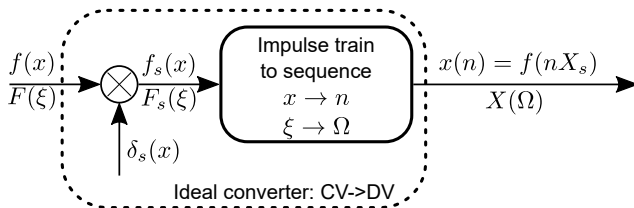
- How much information is lost because of sampling?
- What step should be used?
- What is the relationship between  $F(\xi)$  and  $X(\Omega)$ ?





# Sampling: ideal converter

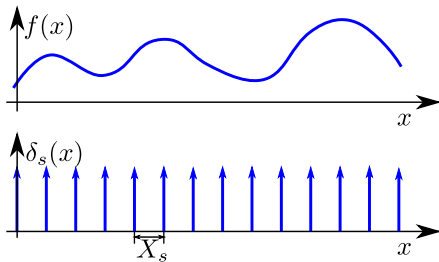
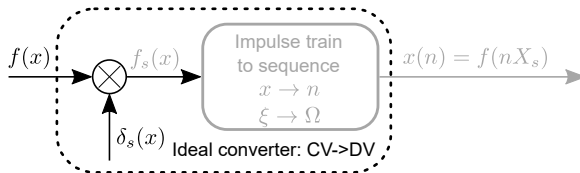
- The following scheme shows an **ideal CV-DV converter**.
- This converter is an **ideal model** useful to understand the sampling process.
  - However, it does not represent a real implementation of an **analog to digital converter** (ADC).



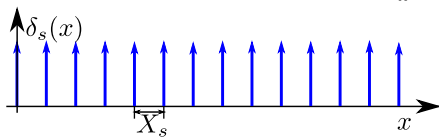
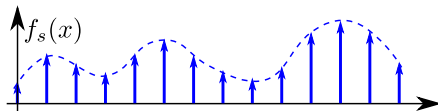
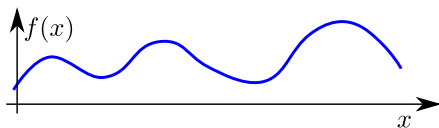
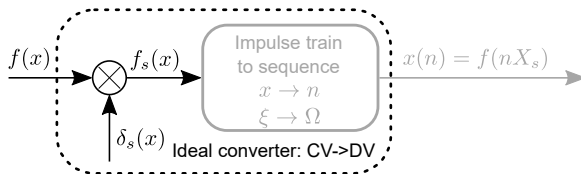
- In the previous picture,  $\delta_s(x) = \sum_{k=-\infty}^{\infty} \delta(x - kX_s)$  represents an impulse train with a sampling rate  $1/X_s$ .



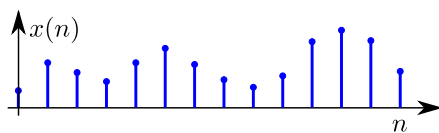
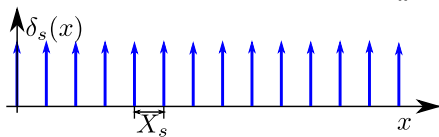
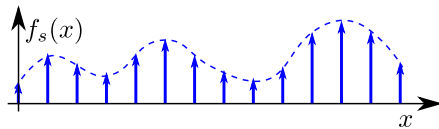
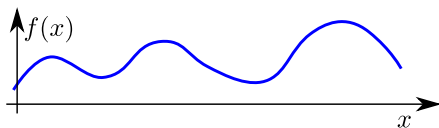
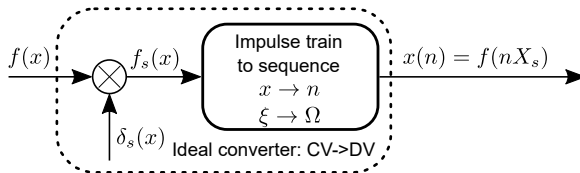
# Sampling: ideal converter



# Sampling: ideal converter



# Sampling: ideal converter



# Sampling: spectrum of the sampled signal

- Taking into account the sampled signal  $f_s(x)$ :

$$f_s(x) = f(x) \sum_{k=-\infty}^{\infty} \delta(x - kX_s)$$

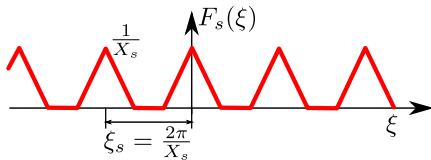
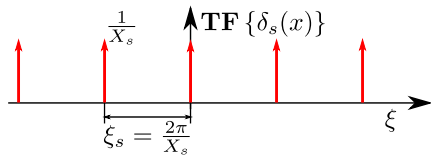
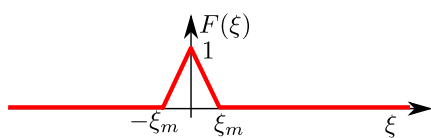
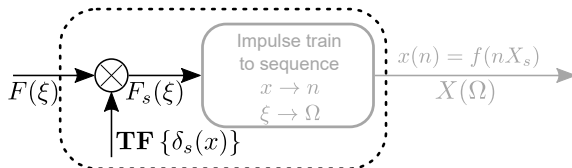
the spectrum is given by:

$$F_s(\xi) = F(\xi) * \left( \frac{1}{X_s} \sum_{k=-\infty}^{\infty} \delta(\xi - k\xi_s) \right) = \frac{1}{X_s} \sum_{k=-\infty}^{\infty} F(\xi - k\xi_s)$$

where the sampling (angular) frequency is defined by  $\xi_s = \frac{2\pi}{X_s}$ .



# Sampling: spectrum of the sampled signal



# Nyquist–Shannon sampling theorem - I

- Considering the previous result, it is clear that a sampled signal  $f_s(x)$  comprises all the signal from the original signal, which could be retrieved by an ideal filter, as long as the sampling frequency  $\xi_s$  is greater than twice the **bandwidth** of the signal  $\xi_m$ .

$$\xi_s > 2\xi_m$$

## Nyquist–Shannon sampling theorem

If a function  $f(x)$  contains no angular frequencies higher than  $\xi_m$ , it is completely determined by giving its ordinates at a series of points spaced  $X_s < \frac{\pi}{\xi_m}$ .



# Nyquist–Shannon sampling theorem - II

- In general, the following quantities are defined:
  - **Nyquist frequency:**  $f_{sN} = \frac{\xi_m}{2\pi}$  is the sampling frequency to suffice to represent  $f(x)$  without losing information.
  - **Nyquist period:**  $X_{sN} = \frac{1}{f_{sN}}$  is the maximum sampling step to suffice to represent  $f(x)$  without losing information.
- Equivalent conditions can be established in terms of the conventional frequency rather than the angular frequency

$$f_s > 2B$$

where  $B$  is the bandwidth of the signal (in terms of conventional frequency instead of angular frequency).





# Relation between the DTFT and the CV FT

- If the spectra of the sampled signal  $f_s(x)$  and the discrete-variable signal  $x(n)$  are compared:

$$f_s(x) = \sum_{n=-\infty}^{\infty} f(nX_s)\delta(x - nX_s) \leftrightarrow F_s(\xi) = \sum_{n=-\infty}^{\infty} f(nX_s)e^{-jnX_s\xi}$$

$$x(n) \leftrightarrow X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$$

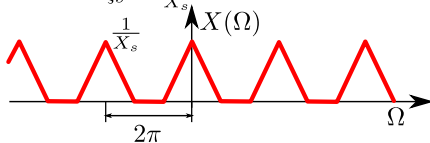
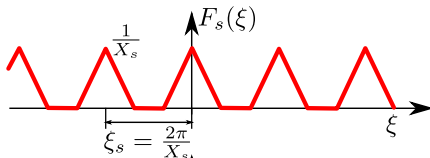
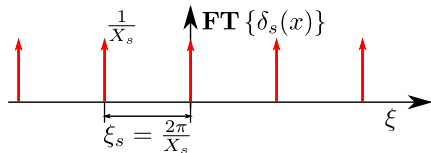
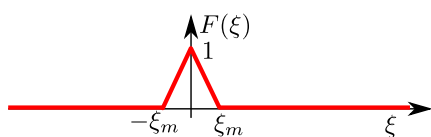
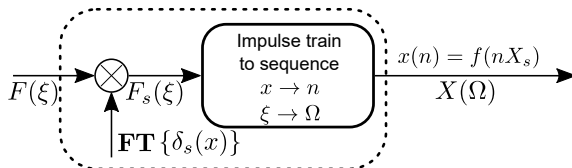
- Hence, both spectra are the same except for an **scaling of the angular frequency by  $X_s$** , i.e.:

$$\Omega = X_s\xi$$

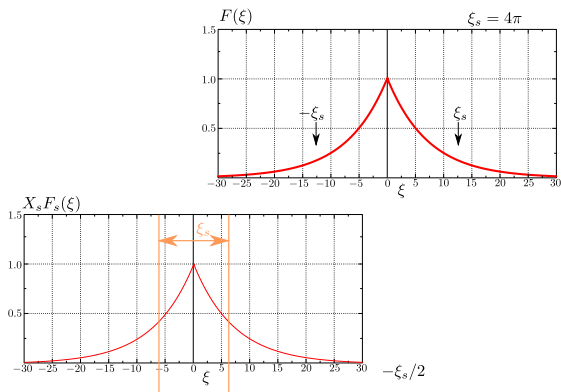
$$X(\Omega) = F_s\left(\xi = \frac{\Omega}{X_s}\right)$$



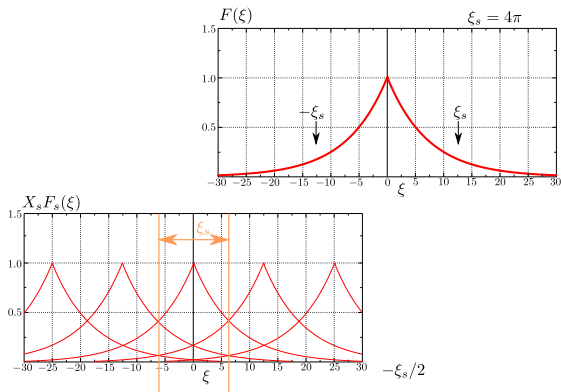
# Relation between the DTFT and the CV FT



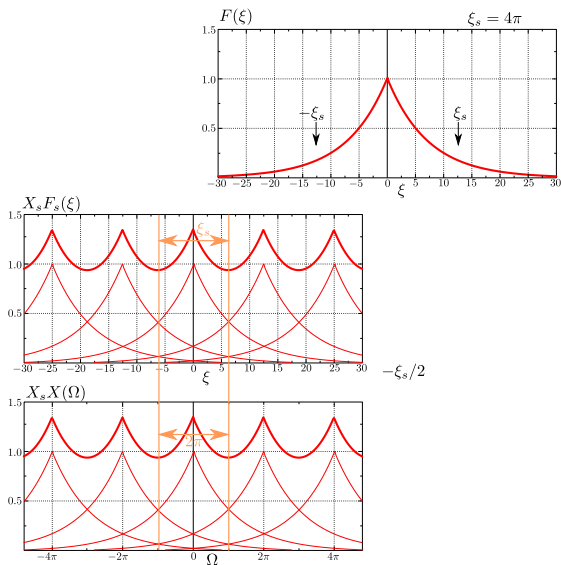
# Non-bandlimited signal



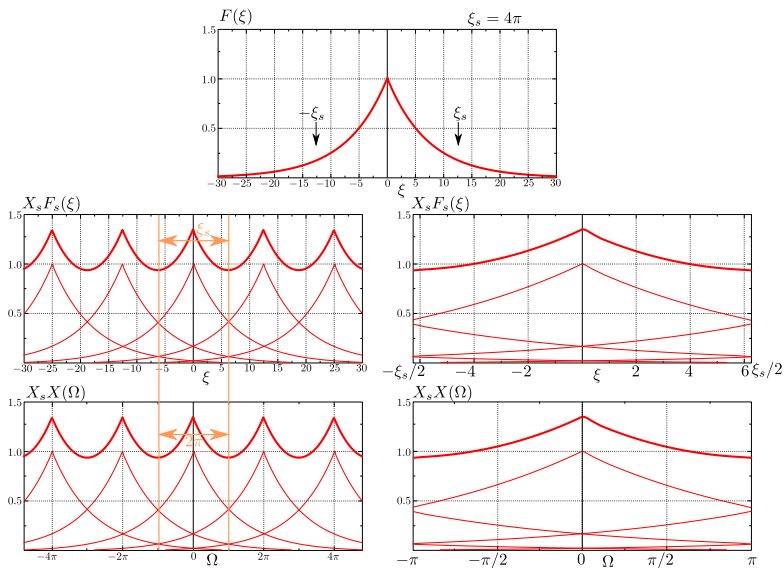
# Non-bandlimited signal



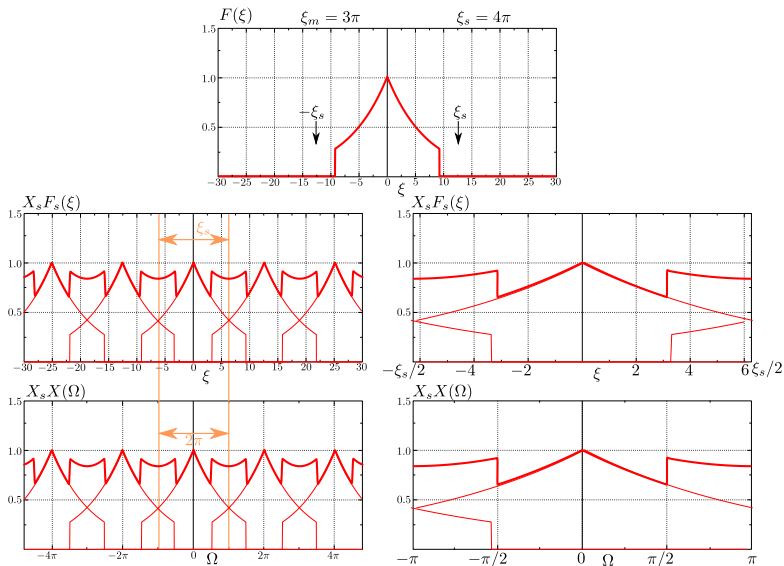
# Non-bandlimited signal



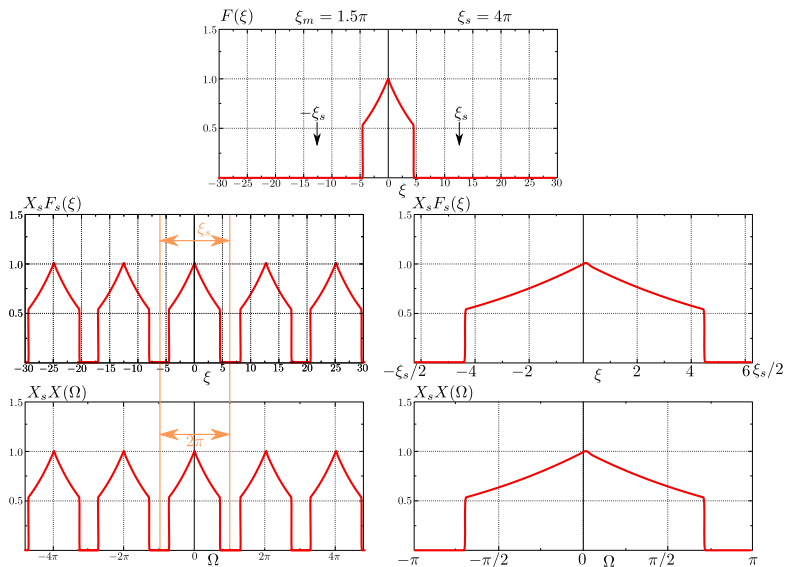
# Non-bandlimited signal



# Band-limited signal - Aliasing



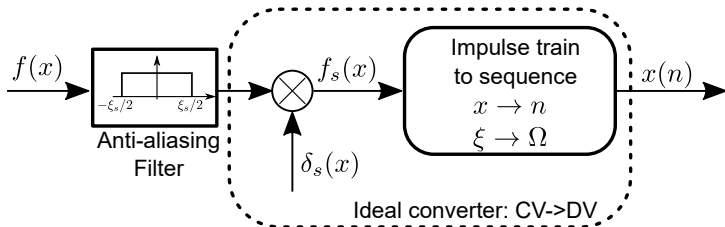
# Band-limited signal - No aliasing





# Anti-aliasing filter

- It is usual to use an **anti-aliasing filter**, which restricts the bandwidth of the input signal.
- It is a low-pass filter with a bandwidth  $\xi_s/2$ , i.e.,  $H(\xi) = \Pi\left(\frac{\xi}{\xi_s}\right)$ .
- Although it **has an impact in the input signal**, it avoids aliasing problems.
  - It use depends on the application.



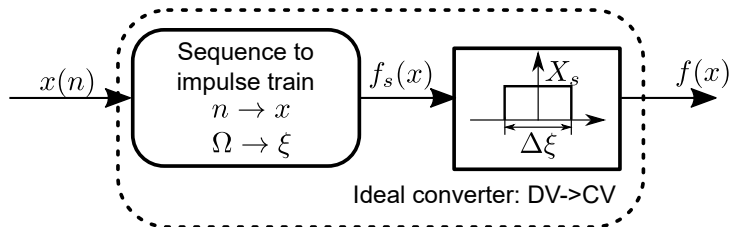
# Discrete-variable to continuous-variable converter - I

- If the sampling frequency is greater than the Nyquist frequency, the signal can be ideally retrieved from  $f_s(x)$  by using a low-pass filter:

$$F(\xi) = F_s(\xi)X_s\Pi\left(\frac{\xi}{\Delta\xi}\right)$$

with  $\Delta\xi = \xi_s$

- Hence, the signal can be recovered from  $x(n)$  using the reverse of the steps of the DV-CV converter: converting from sequence to to impulse train followed by a **low-pass filter** (i.e., reverse steps of a CV to DV converter).



# Discrete-variable to continuous-variable converter - II

- Thus, the reconstructed signal can be expressed as:

$$f_r(x) = X_s \frac{\Delta\xi}{2\pi} f_s(x) * \text{sinc}\left(\frac{\Delta\xi}{2}x\right)$$

wherein the retrieved signal  $f_r(x)$  is equal to the original signal  $f(x)$  if the sampling conditions are met and if  $\xi_m < \frac{\Delta\xi}{2} < \xi_s - \xi_m$ .

## Whittaker–Shannon interpolation formula

A signal  $f(x)$ , whose maximum angular frequency is  $\xi_m$ , sampled at a rate  $\xi_s > 2\xi_m$  can be completely retrieved from its samples by means of the following interpolation formula:

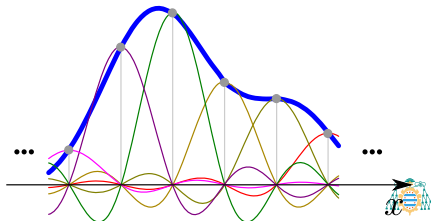
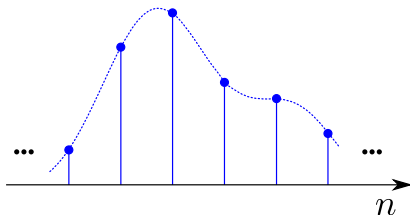
$$f(x) = X_s \frac{\Delta\xi}{2\pi} \sum_{n=-\infty}^{\infty} f(nX_s) \text{sinc}\left(\frac{\Delta\xi(x - nX_s)}{2}\right)$$

# Discrete-variable to continuous-variable converter - III

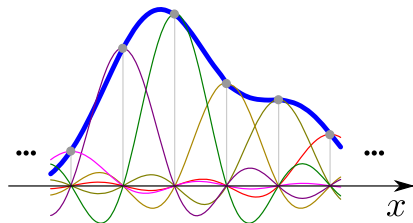
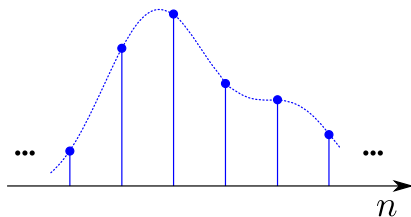
- It is usual to choose a filter with the minimum feasible bandwidth,  $\Delta\xi = \xi_s = \frac{2\pi}{X_s}$ , so that it filters all the unwanted frequencies, which could be contaminated by noise or interferences. In this case:

$$f(x) = \sum_{n=-\infty}^{\infty} f(nX_s) \operatorname{sinc}\left(\pi \frac{(x - nX_s)}{X_s}\right)$$

- It is interesting to note that the **maxima** of each sinc are the **zeros** of the rest of the sincs and, therefore, their value at that point is preserved.



# Discrete-variable to continuous-variable converter - IV



- Note that, by means of sinc interpolation, the original signal is fully reconstructed.
  - Thus, it is said to be an **ideal interpolation filter**.
  - In a practical context, it is not possible to use sinc functions so **suboptimum approaches** are used (zero-order hold, linear interpolation, truncated sincs, etc.)



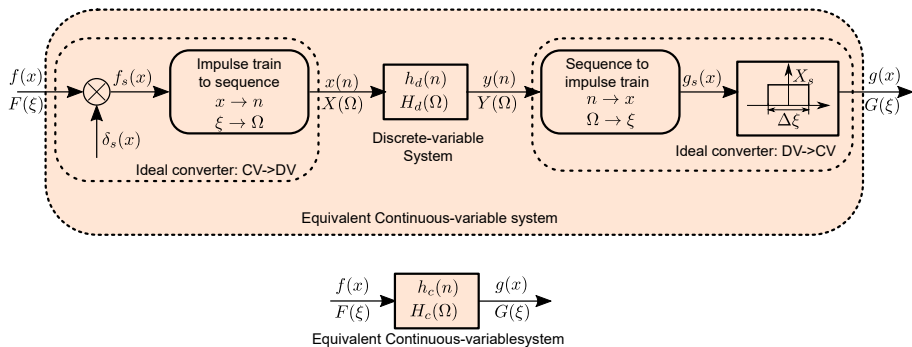
# Equivalent frequencies when using the FFT

- If a signal  $f(x)$  is sampled, it is possible to efficiently estimate its spectrum, which could suffer from aliasing, by means of a FFT.
- If  $N$  samples are available and the acquisition step is  $X_s$ , then the FFT provides frequency samples at:

$$\xi_m = m \frac{2\pi}{N} \frac{1}{X_s}$$



# Modeling a CV LSI system by means of a DV LSI system



- If the Nyquist conditions are met, the DV system is equivalent to the following CV system:

$$H_c(\xi) = H_d(\xi X_s) \Pi\left(\frac{\xi}{\xi_s}\right) = \begin{cases} H_d(\xi X_s) & |\xi| < \xi_s/2 \\ 0 & |\xi| > \xi_s/2 \end{cases}$$



# Modeling a CV LSI system by means of a DV LSI system

## Exercise

We have studied that the product with a signal  $p(x)$  is usually not invariant. Study if the continuous-variable equivalent system is invariant or not and evaluate the requirements.





# Modeling a CV LSI system by means of a DV LSI system

## Exercise

Let  $f(x)$  be a signal whose spectrum is  $F(\xi) = e^{-a|\xi|}$ :

- Calculate the spectrum  $F_s(\xi)$  for an arbitrary sampling rate  $\xi_s$
- Find the signal  $f(x)$  in the real domain.
- Determine the reconstructed signal  $f_r(x)$  in the real domain.
- Repeat the exercise if the signal is band-limited  $F(\xi) = e^{-a|\xi|} \Pi\left(\frac{\xi}{2\xi_m}\right)$

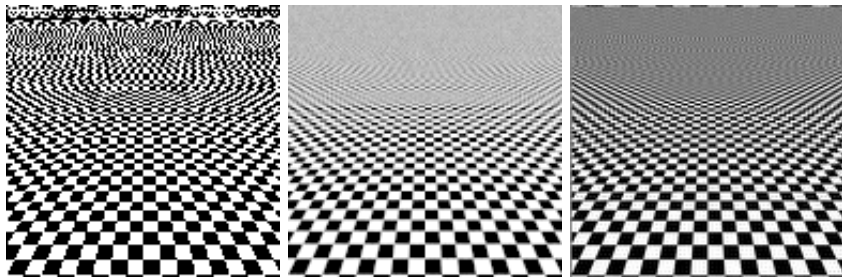
## Solution

See notes in the virtual campus



## Example: Aliased images

Images with fast changes can suffer from aliasing when representing on screens based on discrete samples (i.e., pixels).



Left image shows the aliased image, center image shows the image after a suboptimum filtering (a pulse in the real-domain, i.e., moving average), the right image shows the result after applying an optimum filter (i.e., a sinc).



## Example: stroboscopic effect

Available at <https://www.youtube.com/watch?v=OtxlQTmx1LE>

# Click!



## Example: stroboscopic effect

Available at <https://youtu.be/D1LnySRbO5U?t=5>

# Click!



## Example: recording wheels when spinning

Available at [https://youtu.be/B8EMI3\\_0TO0](https://youtu.be/B8EMI3_0TO0)

# Click!

