Aperiodic signals: Fourier Transform of Discrete-Variable Signals

Chapter 10

Jaime Laviada

Área de Teoría de la Señal y Comunicaciones
Universidad de Oviedo

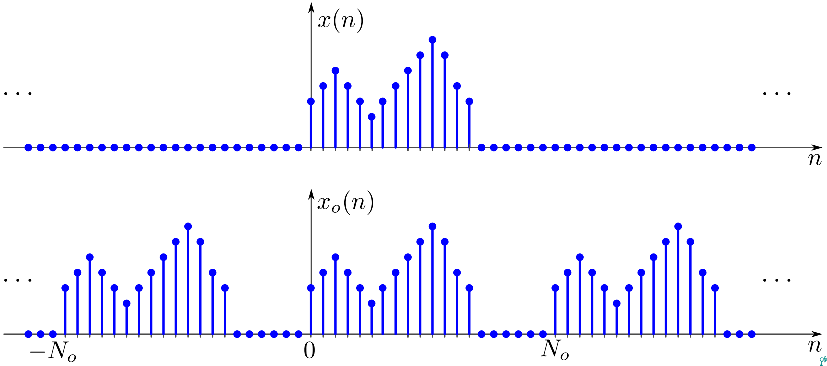
Outline

The Discrete-Variable Fourier Transform

- In an analogous way to continuous-variable aperiodic signals, it is also possible to define a Fourier transform of discrete-variable signals
  - Moreover, it will be also possible to apply this definition to periodic signals with the help of distributions.
- This transform is usually denoted by DTFT (*Discrete-Time Fourier Transform*).
  - This notation can result misleading since it does not always refer to signal in the time-domain.
- The equations supporting this transform can be found using a similar procedure to the used for CV signals (analyzing the limit of the FS as the period approaches infinity).

Development of the DV Fourier Transform - I

- Let us consider a finite-length signal $x(n)$, which can construct a periodic signal $x_0(n)$ for which $x(n)$ is a period.
  - A very similar development can be also carried out for infinite-length signals.



Development of the DV Fourier Transform - II

- Consider the equations of the Fourier series of the periodic function $x_0(n)$, we have:

$$x_0(n) = \sum_{m=0}^{N_0-1} a(m) e^{jm\Omega_0 n}, \quad a(m) = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x_0(n) e^{-jm\Omega_0 n}, \quad \Omega_0 = \frac{2\pi}{N_0}$$

- Since the signal $x(n)$ has a finite length, the previous summation can be written as:

$$a(m) = \frac{1}{N_0} \sum_{n=-\infty}^{\infty} x(n) e^{-jm\Omega_0 n} = \frac{1}{N_0} X(m\Omega_0),$$

wherein the following continuous-variable signal has been defined

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n},$$

It is important to note that this signal is periodic, with period 2π, i.e., $X(\Omega) \in P(2\pi)$.

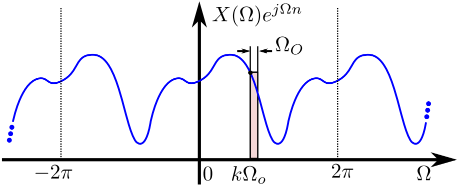
$$X(\Omega + 2\pi) = X(\Omega)$$

Development of the DV Fourier Transform - III

- Replacing the new function in the synthesis equation:

$$x_0(n) = \frac{1}{2\pi} \sum_{m=0}^{N_0-1} X(m\Omega_0) e^{jm\Omega_0 n} \Omega_0.$$

- As in the CV case, the previous summation becomes a Riemann integral over the interval $(0, 2\pi)$ as N_0 approaches infinity.



- Finally, the limit of the previous function results in the original function

$$x(n) = \lim_{N_0 \to \infty} x_0(n) = \frac{1}{2\pi} \int_0^{2\pi} X(\Omega) e^{j\Omega n} d\Omega$$

Analysis and Synthesis equations

- The previous development suggests a new transformation for aperiodic signals.
- This new transformation is the Fourier Transform of discrete-variable signals.
- On one hand, the synthesis equation is given by

$$\boxed{x(n) = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} X(\Omega)\, e^{j\Omega n} d\Omega.}$$

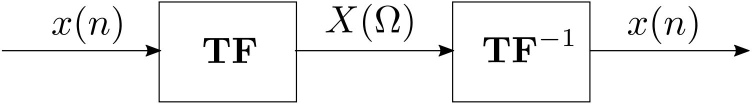
  where an arbitrary integration interval of length 2π has been considered since $X(\Omega)\, e^{j\Omega n}$ is a periodic function, with period 2π.
- On the other hand, the analysis equation is given by

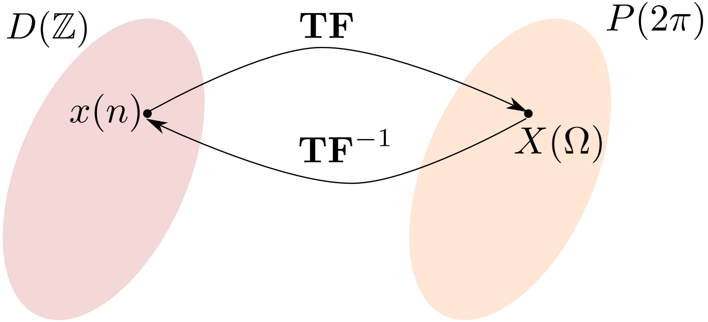
$$\boxed{X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}}$$

Relation to the general theory of transforms

- The previous equations show that, in general, the complex exponentials can be used as basis functions to model DV aperiodic functions
$$\phi(n;\Omega) = e^{j\Omega n}$$
- Thus, the conventional operations for the projection (analysis eq.) and linear combination (synthesis eq.) are
$$X(\Omega) = \langle x(n), \phi(n;\Omega)\rangle$$
$$x(n) = \frac{1}{2\pi}\mathrm{CL}\{X(\Omega)\phi(n;\Omega)\}$$
it is relevant to note that the inner product is defined on $D(\mathbb{Z})$ whereas the linear combination is defined on $P(2\pi)$
- The analysis of the orthogonality and capabilities to generate the signal space $D(\mathbb{Z})$ will be later considered.

Signal spaces





Definition in terms of operators: duality with respect to the FS

- Since $X(\Omega)$ is (continuous-variable) periodic signal, with period $X_o = 2\pi$, it is possible to calculate its FS.
- If the equations of the FS of *continuous-variable* signals are compared with the FT of *discrete*-variable signals, it is possible to represent this by means of:

$$f_0(x) \longleftrightarrow a(m) \text{ FS-CV}$$
$$x(n) = a(-n) \longleftrightarrow X(\Omega) = f_0(\Omega) \text{ FT-DV}$$

- Thus, it is possible to reuse the properties and transform pairs from our previous study on FS of continuous-variable signals.

Criteria for FT convergence

- The analysis equation involves a summation with infinite terms, whose convergence is not always guaranteed.
- In general, the existence of $X(\Omega)$ is sufficient to guarantee the convergence of the inverse transform.

Criteria for FT convergence

- Criterion 1: Absolutely summable sequence

$$\sum_{m=-\infty}^{\infty} |x(n)|^2 < \infty \Rightarrow \exists X(\Omega) \text{such that } \lim_{N\to\infty} d(X(\Omega), X_N(\Omega)) \to 0$$

where $X_N(\Omega) = \mathbf{TF}_N(x(n)) = \sum_{n=-N}^{N} x(n)e^{-j\Omega n}$.

- Criterion 2: Square summable sequence

$$\sum_{m=-\infty}^{\infty} |x(n)| < \infty \Rightarrow |X(\Omega)| < \infty, \forall\Omega \text{ and } x(n) = \mathbf{TF}^{-1}(X(\Omega))$$

The last proof can be found in the appendix. In this case, it is also possible to show that $X(\Omega)$ is a continuous function.

Stable LSI systems and the Fourier Transform

- It is relevant to remember that the impulse response of a stable LSI system is always absolute summable

Corollary

The Fourier Transform of the impulse response of a DV stable LSI system always exits (and it is continuous!).

Properties - I

Periodicity	$X(\Omega) = X(\Omega + 2\pi)$	
Linearity	$\alpha x(n) + \beta y(n)$	$\alpha X(\Omega) + \beta Y(\Omega)$
Shifting	$x(n - n_o)$	$X(\Omega) e^{-j\Omega n_o}$
Multiplication by a complex exp.	$x(n) e^{j\Omega_o n}$	$X(\Omega - \Omega_o)$
Complex conjugate	$x^*(n)$	$X^*(-\Omega)$
Reversal	$x(-n)$	$X(-\Omega)$
Duality with FS	$f_0(x) \longleftrightarrow a(m)$ DSF-VC $x(n) = a(-n) \longleftrightarrow X(\Omega) = f_0(\Omega)$ TF-VD with $X_0 = 2\pi$	
Independent variable scaling	$x(n/p)$	$X(p\Omega)$
Convolution	$x(n) * y(n)$	$X(\Omega) Y(\Omega)$
Multiplication	$x(n) y(n)$	$\frac{1}{2\pi} X(\Omega) * Y(\Omega)$
First difference	$x(n) - x(n-1)$	$\left(1 - e^{-j\Omega}\right) X(\Omega)$
Accumulation	$\sum_{k=-\infty}^{n} x(k)$	$\frac{1}{\left(1 - e^{-j\Omega}\right)} X(\Omega) + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$
Multiplication by n	$n x(n)$	$j \frac{dX(\Omega)}{d\Omega}$
Inner product	$\langle x(n), y(n) \rangle = \frac{1}{2\pi} \langle X(\Omega), Y(\Omega) \rangle$	
Real signals	$x(n) = x^*(n)$	$X(\Omega) = X^*(-\Omega)$
Even signals	$x(n) = x(-n)$	$X(\Omega) = X(-\Omega)$
Odd signals	$x(n) = -x(-n)$	$X(\Omega) = -X(-\Omega)$

Properties- II

- In this particular transform, the following peculiar properties are found:
  - Duality with respect to the FS of CV signals.
  - Periodicity of the spectrum $X(\Omega) = X(\Omega + 2\pi)$
- It is also relevant to note that the FT preserves the inner products and, therefore, the corresponding Parseval's theorem can be formulated:

$$\boxed{E\left[x(n)\right] = \frac{1}{2\pi} E_0\left[X(\Omega)\right]}$$

Properties example: Periodic convolution

- As an example of periodic convolution, let us consider the spectrum of the signal $x(n) = x_1(n)x_2(n)$ with $x_1(n) = \frac{\sin(\pi n/2)}{\pi n}$ and $x_2(n) = \frac{\sin(3\pi n/4)}{\pi n}$

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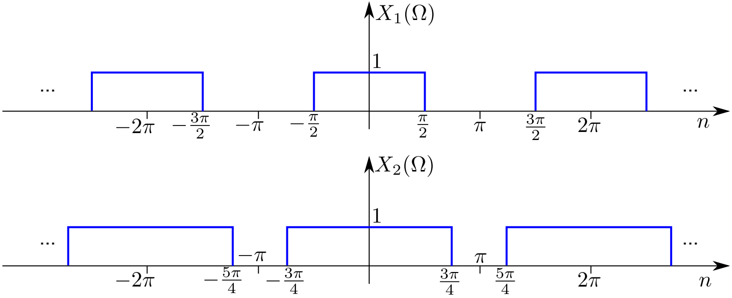

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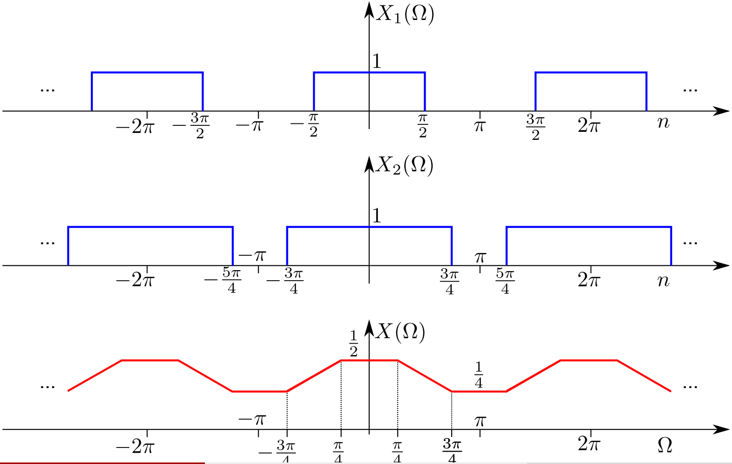


Example of transform pairs - I

$$\boxed{e^{j\Omega_o n} \leftrightarrow 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_o + 2\pi m) = 2\pi\delta_o(\Omega - \Omega_o)}$$

- This pair is easily proved by means of the duality with the FS and the corresponding pair of $\delta_o(x)$.
- Consequently, if $\Omega_o = 0$:

$$1 \leftrightarrow 2\pi\delta_o(\Omega)$$

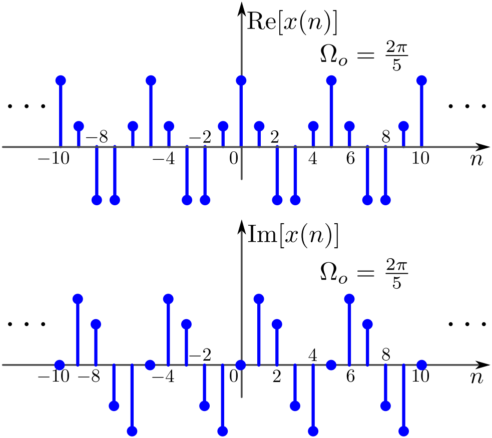
- This transform pair enables us to show the orthogonality of the basis functions:

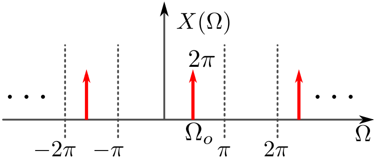
$$\langle \phi(n,\Omega'), \phi(n,\Omega) \rangle = \sum_{n=-\infty}^{\infty} e^{j\Omega' n} e^{-j\Omega n} = \mathbf{TF}(e^{j\Omega' n}) = 2\pi\delta_o(\Omega - \Omega')$$

- In a similar fashion the continuous-variable FT, the basis functions are orthogonal in the sense of the distributions.

Example of transform pairs - I (cont.)

Example with $\Omega_o = \frac{2\pi}{5}$





Example of transform pairs - II

$$\boxed{\frac{\Delta}{2\pi}\text{sinc}\left(n\frac{\Delta}{2}\right) \leftrightarrow \Pi_{0,\Delta}(\Omega)}$$

Proof

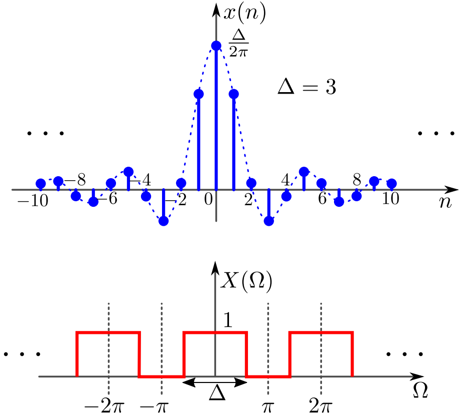
This can also be proved by applying the duality with respect to the continuous-time FS and the corresponding transform pair.

1. Let us consider the pair $\Pi_{0,\Delta}(x) \leftrightarrow \frac{\Delta}{X_O}\text{sinc}\left(\pi m\frac{\Delta}{X_O}\right)$
2. By performing the changes of variable $m = -n$ and setting $X_o = 2\pi$, the result is straightforward.

Exercise: Signal orthogonality

Show the orthogonality of the following signals $x(n) = \frac{1}{2}\text{sinc}\left(n\frac{\pi}{2}\right)$ e $y(n) = \frac{1}{2}\text{sinc}\left(n\frac{\pi}{2}\right)(-1)^n$

Example of transform pairs - II (cont.)



Exercise

Study the spectrum if $\Delta > 2\pi$

Example of transform pairs - III

$$\boxed{\Pi_{2N+1}(n) \leftrightarrow \frac{\sin\left[\left(N+\frac{1}{2}\right)\Omega\right]}{\sin\left(\frac{\Omega}{2}\right)}}$$

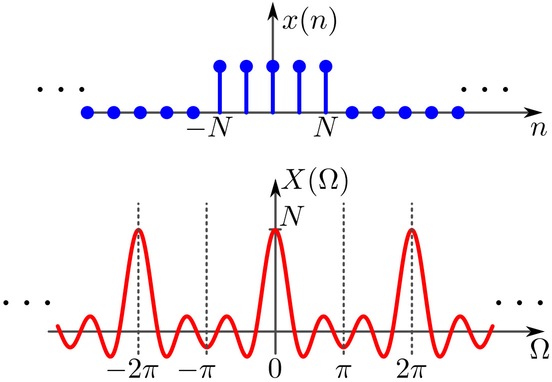
- This function is equal to a *periodic sinc, also known as Dirichlet function*, except for constants.

$$D_N(\Omega) = \begin{cases} \frac{\sin(N\Omega/2)}{N\sin(\Omega/2)} & \Omega \neq 2\pi k \\ (-1)^{k(N-1)} & \Omega = 2\pi k \end{cases}$$

- This function plays a role similar to the conventional sinc and it is key for the interpolation of periodic functions.
- It is possible to show that this function is the result of adding infinite equally-spaced sincs with a spacing of 2π.

Example of transform pairs - III (cont.)

$$\boxed{\Pi_{2N+1}(n) \leftrightarrow \frac{\sin\left[\left(N+\frac{1}{2}\right)\Omega\right]}{\sin\left(\frac{\Omega}{2}\right)}}$$



Example of transform pairs - IV

$$\boxed{a^n u(n) \leftrightarrow \frac{1}{1 - ae^{-j\Omega}}, \, |a| < 1}$$

Proof

Taking into account the definition of the transform:

$$X(\Omega) = \sum_{n=0}^{\infty} a^n e^{-j\Omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\Omega}\right)^n = \frac{1}{1 - ae^{-j\Omega}},$$

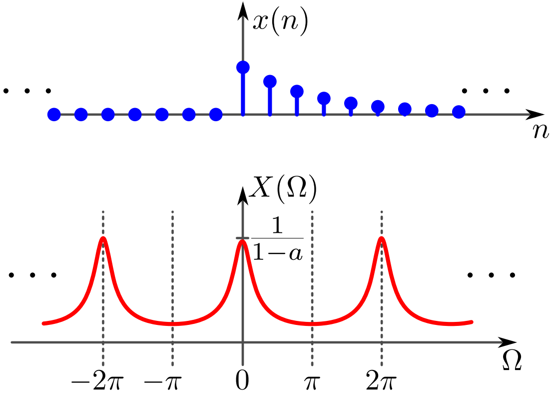
wherein the geometric series formula has been used.

Exercise: Spectrum of the exponential function

Represent the spectrum and identify the maxima/minima position for the following cases $a = 0.7$, $a = -0.7$ y $a = 0.7j$

Example of transform pairs - IV (cont.)

Spectrum of $x(n) = a^n u(n)$ with $a = 0.7$.



Example of transform pairs - IV (cont.)

Exercise

Calculate the Fourier transform of the signal $x(n) = a^{|n|}$ with $|a| < 1$

Solution

$$X(\Omega) = \frac{1}{1-ae^{-j\Omega}} + \frac{ae^{j\Omega}}{1-ae^{j\Omega}} = \frac{1-a^2}{1-2a\cos\Omega+a^2}$$

Example of transform pairs - V

$$\boxed{u(n) \leftrightarrow \frac{1}{1-e^{-j\Omega}} + \pi \sum_{k=-\infty}^{\infty} \delta\left(\Omega - 2\pi k\right)}$$

Proof

The proof can be easily accomplished by taking into account the close relation with $\delta(n)$:

$$u(n) - u(n-1) = \delta(n) \Rightarrow \mathbf{TF}\left(u(n)\right)\left(1 - e^{-j\Omega}\right) = 1$$

in addition, we must also take into account the possibility of general solutions to the previous equation by including distribution. Thus, since $\left(1 - e^{-j\Omega}\right)$ is zero at $\Omega = 2\pi k$:

$$\mathbf{TF}\left(u(n)\right) = \frac{1}{\left(1-e^{-j\Omega}\right)} + C \sum_{k=-\infty}^{\infty} \delta\left(\Omega - 2\pi k\right),$$

where C is a constant to be found. (cont.)

Example of transform pairs - V (cont.)

Proof (cont.)

The constant C can be found by observing the FT of an odd signal constructed from $u(n)$, for example:

$$x(n) = u(n) - \frac{1}{2} - \frac{1}{2}\delta(n)$$

whose FT is:

$$X(\Omega) = \frac{1}{\left(1 - e^{-j\Omega}\right)} + C\sum_{k=-\infty}^{\infty}\delta(\Omega - 2\pi k) - \pi\sum_{k=-\infty}^{\infty}\delta(\Omega - 2\pi k) - \frac{1}{2}$$

since the FT must be also odd and the term $\frac{1}{\left(1-e^{-j\Omega}\right)} - \frac{1}{2} = \frac{1+e^{-j\Omega}}{2\left(1-e^{-j\Omega}\right)}$ is already even, then the deltas must cancel each other out, then:

$$C - \pi = 0 \Rightarrow C = \pi$$

Example of transform pairs - VI

- Periodic signal:

$$\boxed{x_o(n) = \sum_{m=0}^{N_o-1} a(m) e^{jm\Omega_o n} \leftrightarrow 2\pi \sum_{m=-\infty}^{\infty} a(m)\delta(\Omega - m\Omega_o)},$$

it is relevant to note that the summation interval has been extended by exploiting the periodicity of the coefficients $a(m)$.

Proof

$$\sum_{m=0}^{N_o-1} a(m) e^{jm\Omega_o n} \leftrightarrow 2\pi \sum_{m=0}^{N_o-1} a(m) \sum_{k=-\infty}^{\infty} \delta(\Omega - m\Omega_o + 2\pi k) =$$

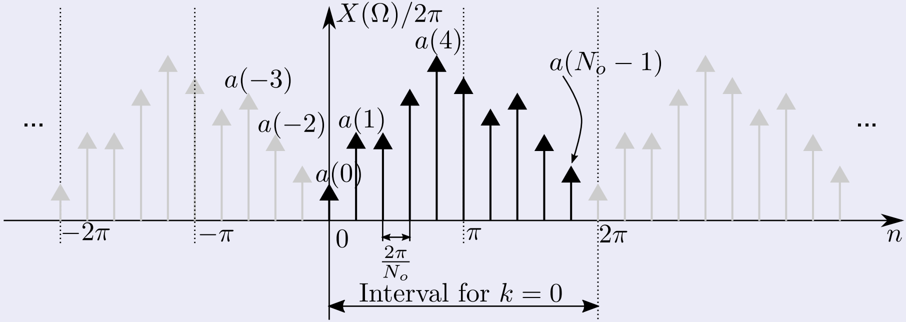
$$2\pi \sum_{k=-\infty}^{\infty} \sum_{m=0}^{N_o-1} a(m)\delta(\Omega - m\Omega_o + 2\pi k) = 2\pi \left(\sum_{m=0}^{N_o-1} a(m)\delta(\Omega - m\Omega_o) \right) * \left(\sum_{k=-\infty}^{\infty} \delta(\Omega + 2\pi k) \right)$$

(cont.))

Example of transform pairs - VI (cont.)

Proof (cont.)

- Taking into account the previous development, the FT of a periodic signal can be represented as below. The inspection of the figure together with the periodicity of $a(m)$, validates the transform pair.



Transform pairs of aperiodic functions

Delta	$\delta(n)$	1
Shifted Delta	$\delta(n-n_o)$	$e^{-j\Omega n_o}$
Unit step	$u(n)$	$\frac{1}{1-e^{-j\Omega}}+\pi\sum_{m=-\infty}^{\infty}\delta\left(\Omega-2\pi m\right)$
Exponential with imaginary exponent	$e^{j\Omega_o n}$, with $\Omega_o/2\pi$ irrational	$2\pi\sum_{m=-\infty}^{\infty}\delta\left(\Omega-\Omega_o-2\pi m\right)$
Unit pulse with width $2N+1$	$\Pi_{2N+1}(n)$	$\frac{\sin\left[\left(N+\frac{1}{2}\right)\Omega\right]}{\sin\left(\frac{\Omega}{2}\right)}$
Exponencial functions	$a^n u(n)$, $\lvert a\rvert<1$	$\frac{1}{1-ae^{-j\Omega}}$
	$(n+1)\,a^n u(n)$, $\lvert a\rvert<1$	$\frac{1}{\left(1-ae^{-j\Omega}\right)^2}$
	$\frac{(n+p-1)!}{n!(p-1)!}a^n u(n)$, $\lvert a\rvert<1$	$\frac{1}{\left(1-ae^{-j\Omega}\right)^p}$

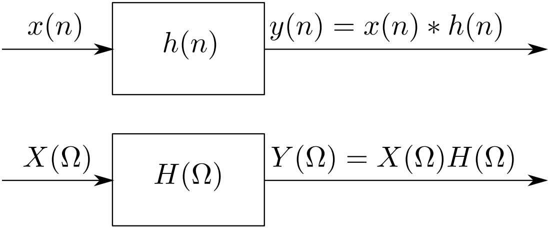
Transform pairs of periodic functions

Exponential with imaginary exponent	$e^{jk\Omega_o n}$	$2\pi \sum_{m=\infty}^{\infty} \delta\left(\Omega - k\Omega_o + 2\pi m\right)$
Cosine	$\cos\left(k\Omega_o n\right)$	$\pi \sum_{m=\infty}^{\infty} \delta\left(\Omega - k\Omega_o + 2\pi m\right) + +\pi \sum_{m=\infty}^{\infty} \delta\left(\Omega + k\Omega_o + 2\pi m\right)$
Sine	$\sin\left(k\Omega_o n\right)$	$\frac{\pi}{j} \sum \delta\left(\Omega - k\Omega_o + 2\pi m\right) - -\frac{\pi}{j} \sum_{m=\infty}^{\infty} \delta\left(\Omega + k\Omega_o + 2\pi m\right)$
Constant	K	$2\pi K \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)$
FS	$\sum_{m=\langle N_o \rangle} a(m) e^{jm\Omega_o n}$	$2\pi \sum_{m=-\infty}^{\infty} a(m)\delta(\Omega - m\Omega_o)$
Dirac comb	$\delta_o(n)$	$\frac{2\pi}{N_o} \sum_{m=-\infty}^{\infty} \delta(\Omega - m\Omega_o)$
Pulse train	$\Pi_{o,2N+1}(n)$	FS with $a(m) = \frac{1}{N_o} \frac{\sin\left[\left(N+\frac{1}{2}\right)m\Omega_o\right]}{\sin\left(\frac{m\Omega_o}{2}\right)}$

Connection between FT and LSI systems

- In a similar fashion to the continuous case, the convolution property enable us to determine the spectrum of the output signal by means of a simple multiplication in the spectral domain

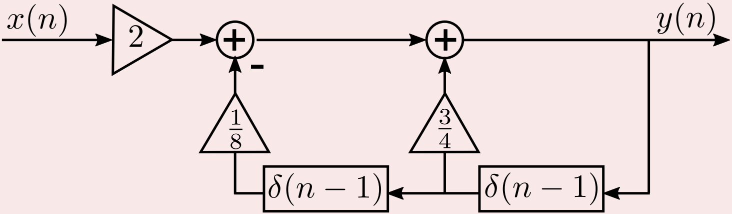
$$x(n) * h(n) \leftrightarrow X(\Omega)H(\Omega)$$



LSI systems and the Fourier transform

Exercise

Consider the following system:



1. Determine the difference equation that models the system.
2. Find the transfer function.
3. Calculate the impulse response in the real-domain.
4. Determine the output of the system if the input is $x(n) = \left(\frac{1}{4}\right)^n u(n)$.

Connection between the discrete-variable FS and the TF

- The coefficients of the FS of a periodic signal $x_o(n)$ were defined as:

$$a(m) = \frac{1}{N_o} \sum_{n=0}^{N_o-1} x_o(n) e^{-jm\Omega_o n}$$

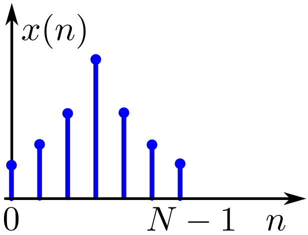
- If $x(n)$ represents a signal constructed as a period of the signal from 0 to $N_0 - 1$, then

$$a(m) = \frac{1}{N_o} \sum_{n=0}^{N_o-1} x(n) e^{-jm\Omega_o n} = \frac{1}{N_o} \sum_{n=-\infty}^{\infty} x(n) e^{-jm\Omega_o n}$$

- Comparing the previous expression with the FT, it is possible to conclude that

$$\boxed{a(m) = \frac{1}{N_o} X(m\Omega_o)}$$

Connection between the discrete-variable FS and the TF



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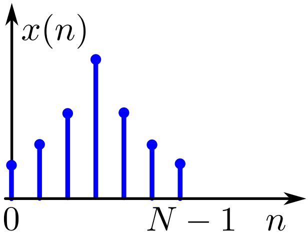
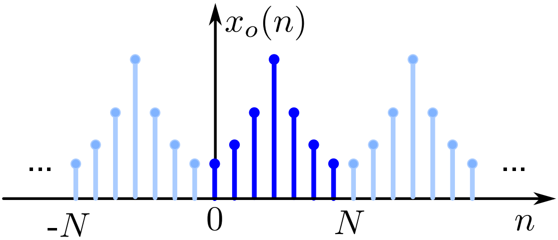
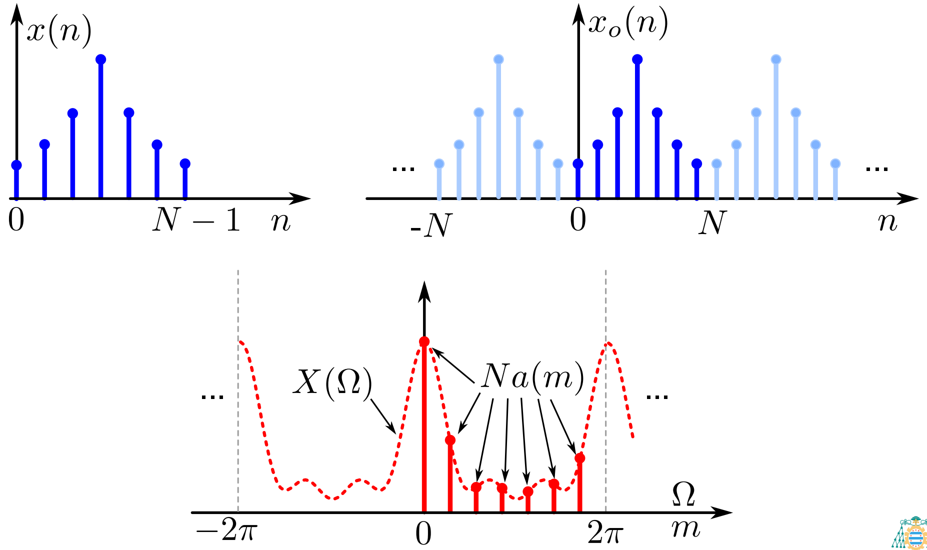




Connection between the discrete-variable FS and the TF



The Discrete Fourier Transform - I

Sampling the spectrum of the FT

A signal with length N can be characterized by means of only N samples in the spectral domain.

- The previous fact paves the way to the numerical calculation of Fourier Transforms.
  - In practice, computing instruments (computers, microcontrollers, etc.) cannot handle a continuous spectrum.
  - The conventional approach is to use samples of the signal and their spectrum
  - The Discrete Fourier Transform is able to work with finite discrete-variable signals and with a finite number of spectrum samples, which fully model the spectrum.
- In this context, the Discrete Fourier Transform is defined.
- From an operational (but not conceptual) point of view, it is equivalent to the FS of discrete-variable signal.
- It can only be applied to finite-length signals.

The Discrete Fourier Transform - II

Analysis and synthesis equations for the DFT

$$\boxed{X_d(m) = \sum_{n=0}^{N-1} x(n) e^{-jm\Omega_o n}}$$

$$\boxed{x(n) = \frac{1}{N} \sum_{n=0}^{N-1} X_d(m) e^{jm\Omega_o n}}$$

where $\Omega_o = \frac{2\pi}{N}$.

- Note that the factor $1/N$ has been moved from the analysis equation to the synthesis equation so the DFT samples are equal to the samples of the DTFT

$$X_d(m) = X(m\Omega_o)$$

Notation

- In a similar fashion to the rest of transforms, there is a large number of alternative definitions, which can be preferred depending on the context:
  - The book *Numerical Recipes* use the opposite convention for the sign of the complex exponentials.
  - The library *FFTW* drop the normalization factor $1/N$.
  - Matlab uses the FFTW... but it includes the previous factor in the inverse transform.
  - The books "Signals and Systems" and "Digital Signal Processing", both by Prof. Alan Oppenheim, use different definitions of the discrete-variable FS, which are later use to define the DFT.

Spectrum interpolation by means of the FT

- It is often to need some additional samples of the spectrum.
- If the length of the signal is odd $N = 2M + 1$, the signal over a period can be represented as:

$$x(n) = x_o(n)\Pi_{2M+1}(n - M)$$

- Thus, the spectrum of the signal can be represented by a *periodic convolution*:

$$X(\Omega) = \frac{1}{2\pi}\left(\frac{2\pi}{N}\sum_{m=-\infty}^{\infty} X\left(m\Omega_o\right)\delta(\Omega - m\Omega_o)\right) * \left(\frac{\sin\left[\frac{N}{2}\Omega\right]}{\sin\left(\frac{\Omega}{2}\right)} e^{-j\Omega\frac{N-1}{2}}\right)$$

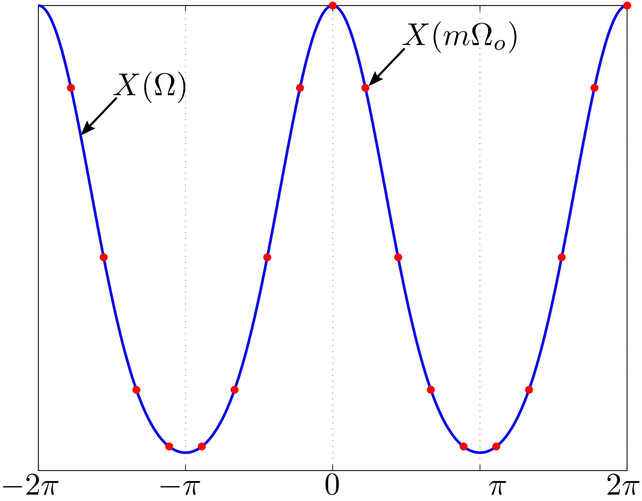
- Thus, it is possible to perfectly retrieve the spectrum of the signal by means of the following equation:

$$\boxed{X(\Omega) = \sum_{m=\langle N\rangle} X\left(m\Omega_o\right) D_N\left(\Omega - m\Omega_o\right) e^{-j(\Omega - m\Omega_o)\frac{N-1}{2}}}$$

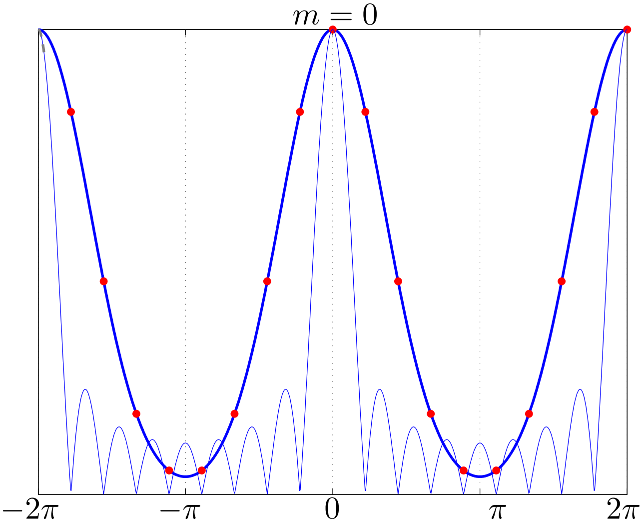
wherein $D_N(\Omega)$ is the Dirichlet function or periodic sinc.

- A similar development can be performed for signal whose length is even.

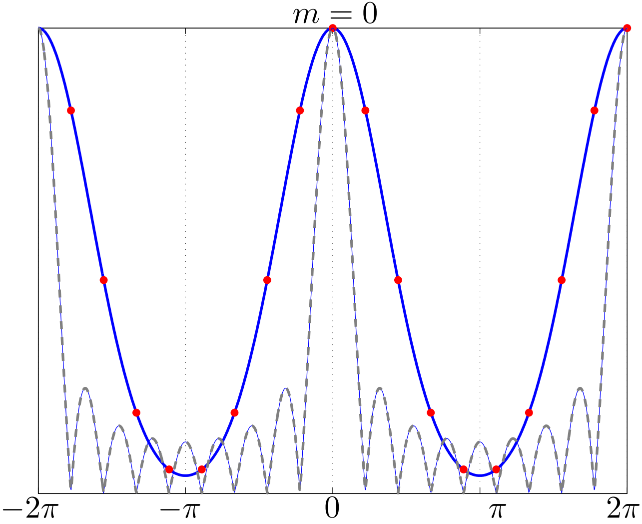
Spectrum interpolation by means of the FT



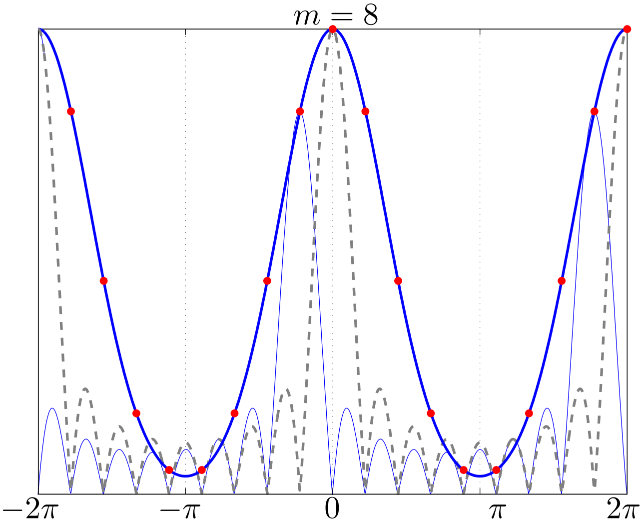
Spectrum interpolation by means of the FT



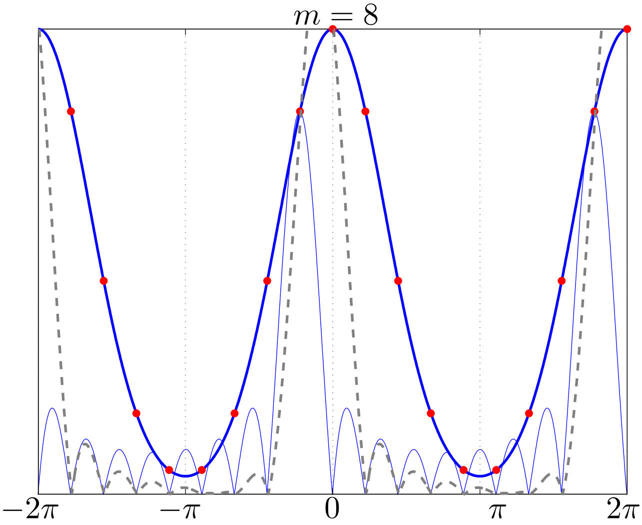
Spectrum interpolation by means of the FT



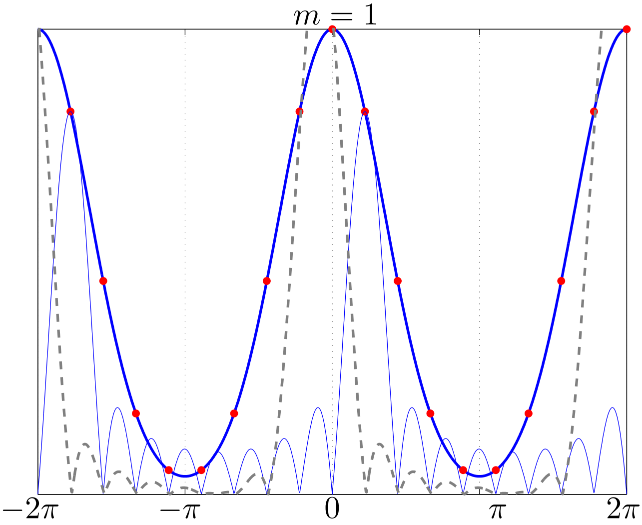
Spectrum interpolation by means of the FT



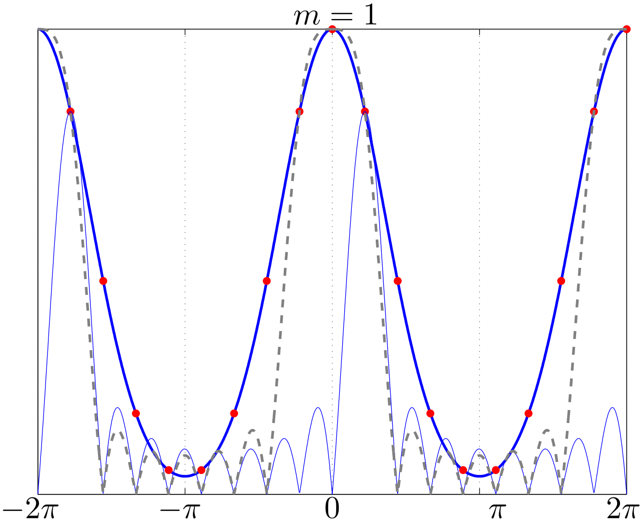
Spectrum interpolation by means of the FT



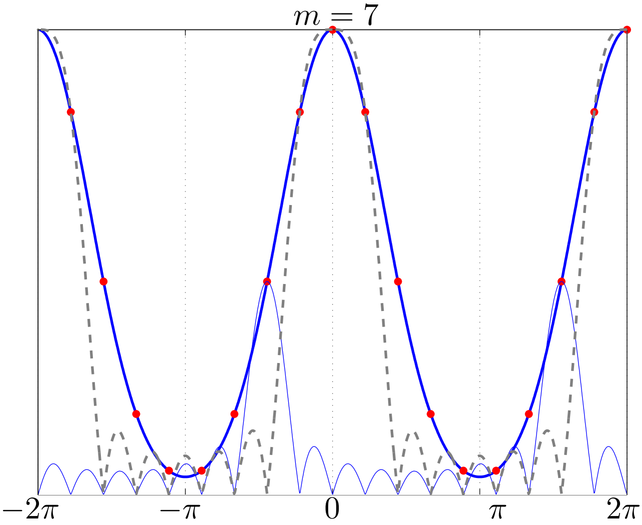
Spectrum interpolation by means of the FT



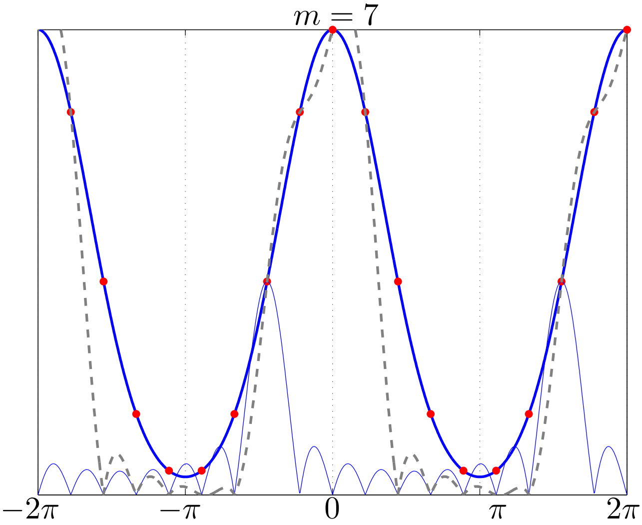
Spectrum interpolation by means of the FT



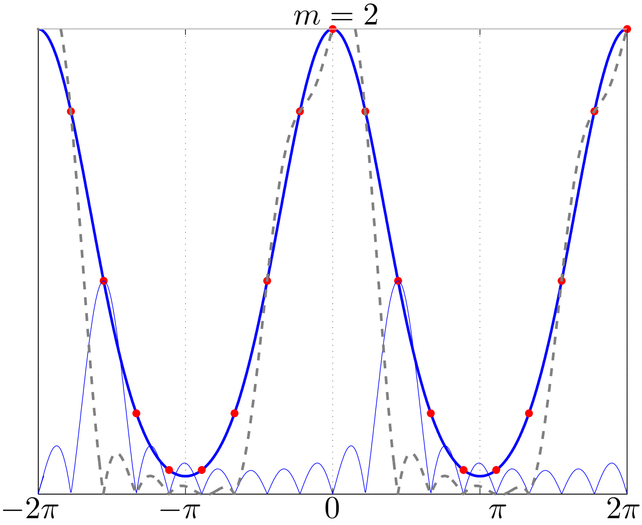
Spectrum interpolation by means of the FT



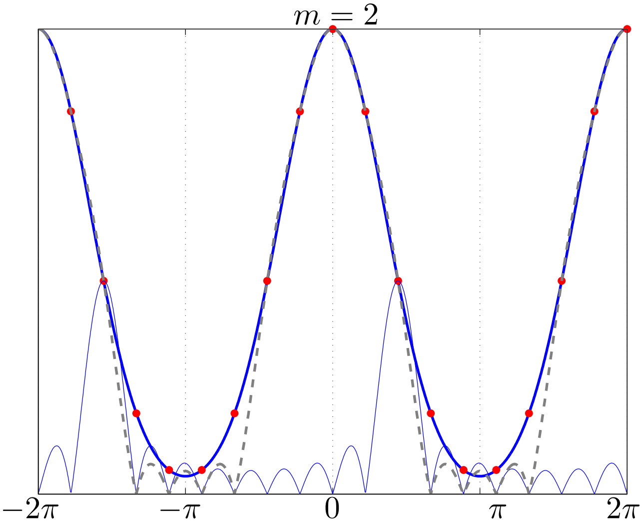
Spectrum interpolation by means of the FT



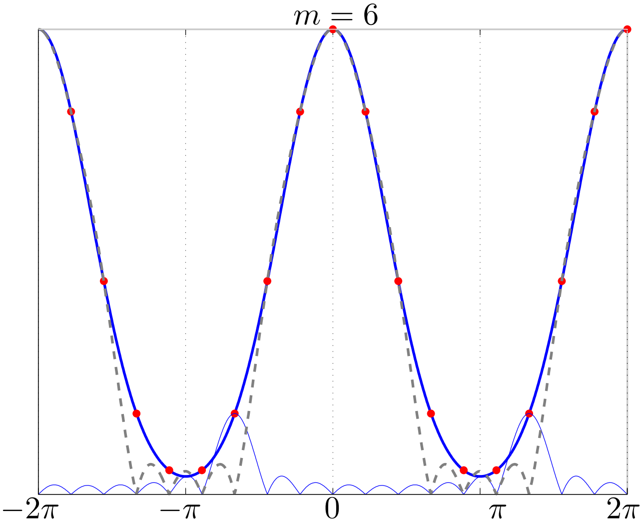
Spectrum interpolation by means of the FT



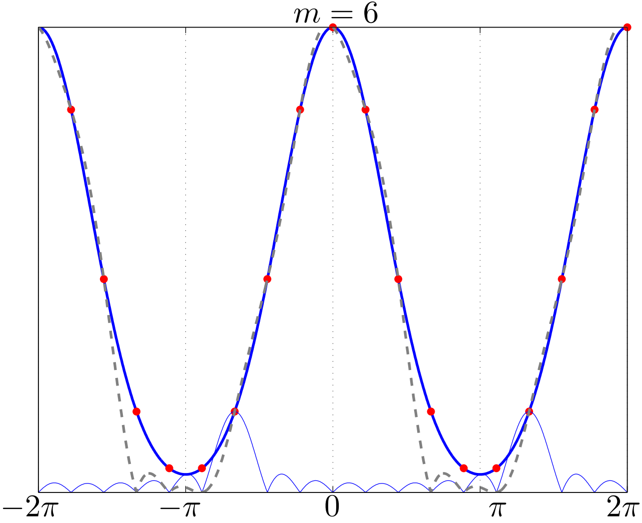
Spectrum interpolation by means of the FT



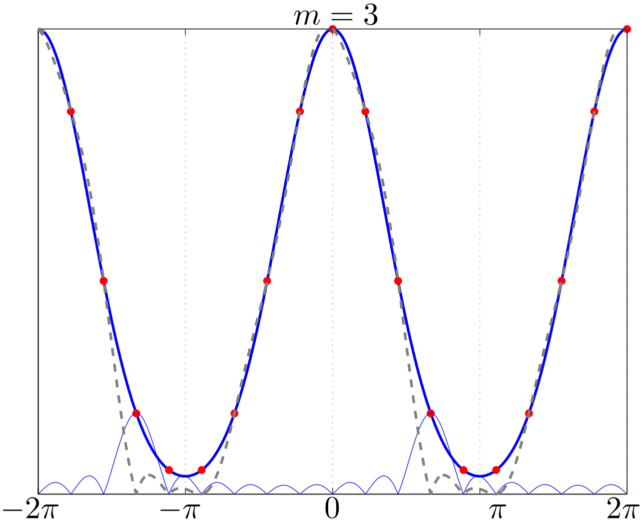
Spectrum interpolation by means of the FT



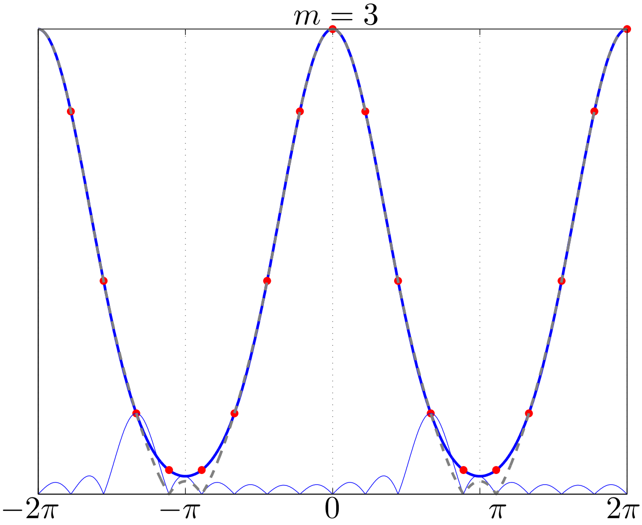
Spectrum interpolation by means of the FT



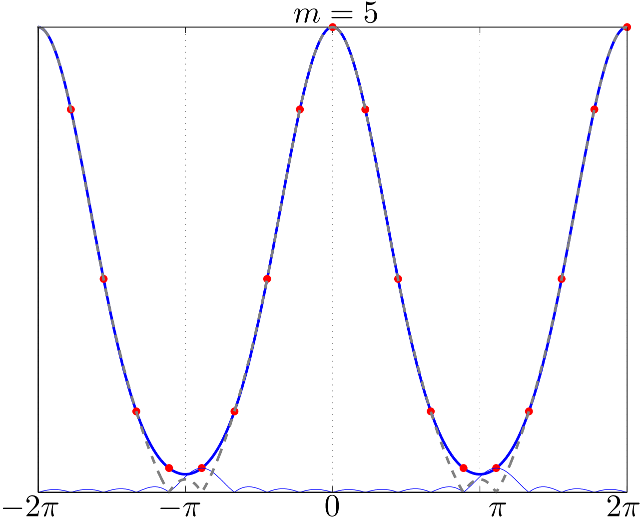
Spectrum interpolation by means of the FT



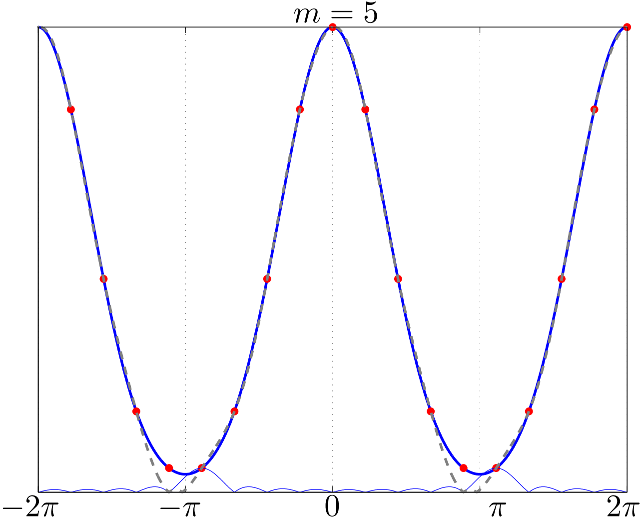
Spectrum interpolation by means of the FT



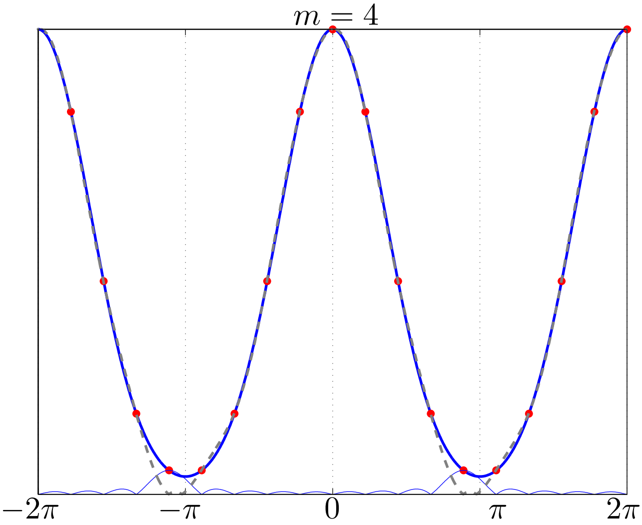
Spectrum interpolation by means of the FT



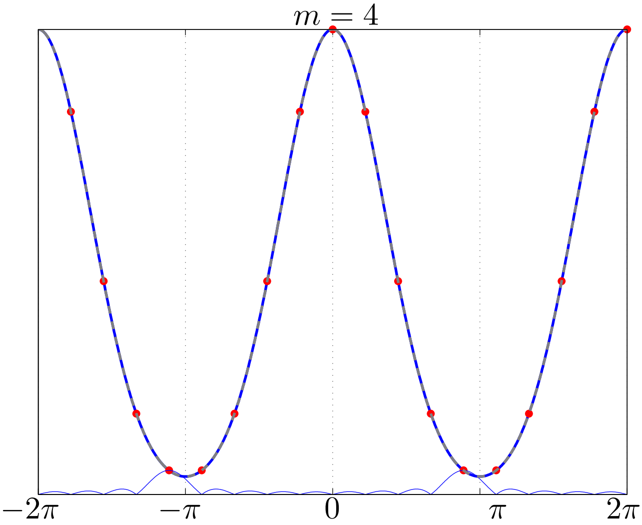
Spectrum interpolation by means of the FT



Spectrum interpolation by means of the FT



Spectrum interpolation by means of the FT



Zero padding - I

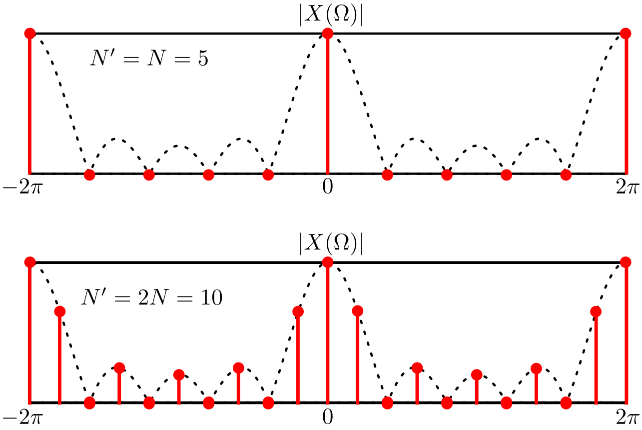
- The DFT enable us to obtain N samples of the spectrum of a signal with a length N.
- It is possible to obtain more spectrum sample by means of the previous interpolation. However, the interpolation requires a larger computational burden.
- A more efficient way is to resort to zero padding:
  - This technique is based on padding the signal with r M zeros at the end of the sequence resulting in a new sequence $x'(n)$ with length $N' = N + M$
  - The corresponding DFT is :

$$X_d'(m) = \sum_{n=0}^{N'-1} x'(n) e^{-jm\Omega_o' n} = \sum_{n=0}^{N-1} x(n) e^{-jm\frac{2\pi}{N'} n}$$

  - Note that $X_d'(m) = X(m\frac{2\pi}{N'})$
- Thus, after padding the signal, the DFT of the new sequence also corresponds to spectral samples of $x(n)$ but with a shorter distance between them

Zero padding - II

Example of calculation of spectral samples of the signal $\Pi_{2N+1}(n-N)$ with $2N+1=5$



Zero padding - III: Interpolation

- This approach is also useful to perform an efficient interpolation of signals characterized by a set of equally-spaced samples.
- The approach comprises three steps DFT+Zero padding+IDFT:
  - The signal cannot be a spectrum signal. Although the results will not be perfect, it can be any arbitrary signal.
  - As expected, the quality of the interpolation depends on the number of samples and the distance between them.
- The interpolation is performed in terms of a periodic sinc (i.e., Dirichlet function) and it assumes that the signal is periodic.

The Fast Fourier Transform - *FFT*

- The computational complexity of directly calculating a DFT is $O\left(N^2\right)$ because it requires a matrix-vector product.
- There a number of algorithms taking advantage of the nature of the basis functions in order to reduce the computational complexity to

$$\boxed{O(N \log N)}$$

- These algorithms are known as *Fast Fourier Transform (FFT)* and they are particularly efficient if N is a power of 2.
  - It is usual to pad a signal with zeros so its length becomes a power of 2.
- The use of the FFT is so widespread that it is usual to refer to the DFT (or even the FT) as FFT.

Periodic and aperiodic convolutions by means of the FFT

- The computational complexity of the convolution is $O(N^2)$.
- Taking into account the complexity of the FFT, it is preferred to calculate a periodic convolution by means of the following workflow:
  1. Compute the FFT ($O(N \log N)$),
  2. Multiplication of the spectra ($O(N)$)
  3. Compute the inverse FFT ($O(N \log N)$)

Exercise: aperiodic convolution by means of the FFT

Propose a strategy to compute the aperiodic convolution by means of the FFT.

Appendix

Proof of the second convergence criterion

- The second convergence criterion will be proved next:

$$|X(\Omega)| = \left| \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \right| \leq \sum_{n=-\infty}^{\infty} |x(n)| \left| e^{-j\Omega n} \right| = \sum_{n=-\infty}^{\infty} |x(n)|$$

If the analysis equation is inserted into the synthesis equation, we have:

$$\hat{x}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{m=-\infty}^{\infty} x(n) e^{-j\Omega m} \right) e^{j\Omega n} d\Omega$$

- Since the signal is absolutely summable, then the Fubini-Tonelli theorem can be applied and, therefore, the integral and the summation order can be exchanged

$$\hat{x}(n) = \sum_{m=-\infty}^{\infty} x(n) \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\Omega(n-m)} d\Omega \right) = \sum_{m=-\infty}^{\infty} x(n) \delta(m-n) = x(n)$$

wherein the result $\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\Omega(n-m)} d\Omega = \delta(m-n)$ is straightforward.