# Periodic signals: Fourier Series of discrete-variable signals

Chapter 9

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# Outline

1. Complex exponential as basis functions
   - Overview of the algebra of the signal space
   - Basis functions
   - Properties of the basis functions
   - Analysis and Synthesis Equations
2. Convergence of the FS
3. Properties
   - Properties
4. Example of FS
5. Connection between FS and LSI systems

# Algebra of the Signal Space

- In this chapter, the signal space $D(N_0)$ is considered. In this space, the following operations are defined:

$$\langle x_0(n), y_0(n)\rangle = \sum_{n=\langle N_0\rangle} x_0(n) y_0^*(n)$$

$$\|x(n)\|^2 = \langle x_0(n), x_0(n)\rangle$$

$$d\left(x_0(n), y_0(n)\right) = \|x_0(n) - y_0(n)\|$$

- In addition, the periodic convolution of discrete-variable signals has been defined as:

$$x_0(n) * y_0(n) = \sum_{k=\langle N_0\rangle} x_0(k) y_0(n-k)$$

# Basis functions

- Since the dimension of the signal space is $N_o$, the bases will be finite and numerable. The employed basis will be denoted by $\phi(n;m)$.
- In this case, the transform (basis change) can be described as:

$$a(m) = \frac{1}{\|\phi(n;m)\|^2} \langle x_0(n), \phi(n;m) \rangle$$

$$x_0(n) = \sum_{n=0}^{N_0-1} a(m)\phi(n;m).$$

- Later, it will be shown that the complex exponentials are a basis of the DV periodic signals:

$$\phi(n;m) = e^{jm\Omega_0 n} \text{ where } \Omega_0 = \frac{2\pi}{N_0}.$$

- It is relevant to note that the basis functions are periodic along $n$ as well as $m$

$$\phi(n;m) = \phi(n;m+N_0), \quad \phi(n;m) = \phi(n+N_0;m)$$

# Basis orthogonality

- Let's check if the previous DV functions are orthogonal:

$$\langle \phi(n;m_1), \phi(n;m_2)\rangle = \sum_{n=0}^{N_0-1} e^{j(m_1-m_2)\frac{2\pi}{N_0}n} = \sum_{n=0}^{N_0-1} \left(e^{j(m_1-m_2)\frac{2\pi}{N_0}}\right)^n =$$

$$= \begin{cases} N_0 & \text{if } m_1-m_2 = kN_0 \\ \frac{1-e^{j(m_1-m_2)\frac{2\pi}{N_0}N_0}}{1-e^{j(m_1-m_2)\frac{2\pi}{N_0}}} = 0 & \text{if } m_1-m_2 \neq kN_0 \end{cases} \forall k \in \mathbb{Z}$$

- Since the basis functions are orthogonal, they are also linearly independent. Moreover, since the dimension of the vector space is $N_0$, then:

**Basis functions**

The functions $\phi(n;m) = e^{jm\Omega_0 n}$, where $\Omega_0 = \frac{2\pi}{N_0}$, are a basis of the *entire* signal space $D(N_0)$.

# Analysis and Synthesis Equations

## Analysis and Synthesis Equations

The equations defining the FS in the case of discrete-variable signals are:

$$\boxed{a(m) = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} x_0(n) e^{-jm\Omega_0 n}}$$

$$\boxed{x_0(n) = \sum_{m=\langle N_0 \rangle} a(m) e^{jm\Omega_0 n}}$$

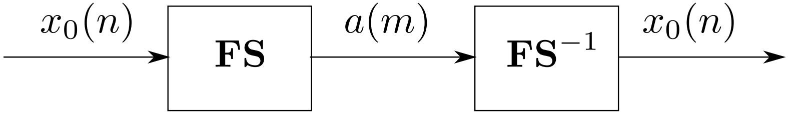
where $\Omega_0 = \frac{2\pi}{N_0}$

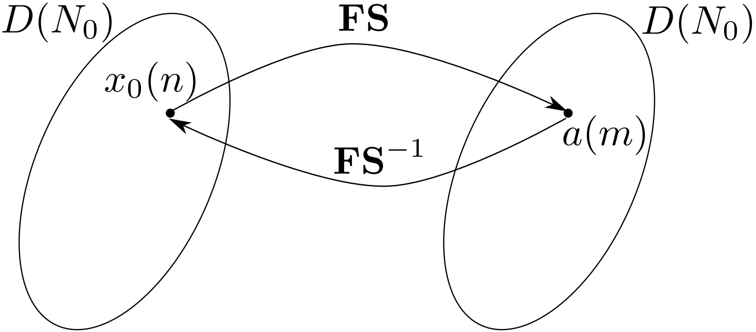
- It is relevant to note that, according to the properties of the basis functions, the coefficients of the FS are also periodic

$$a(m) = a(m + N_0)$$

and, therefore, $N_0$ consecutive coefficients are enough to define the FS (and, consequently, the signal).

# Signal space



# Criteria for the convergence

# General properties

| | | |
|---|---|---|
| Linearity | $\alpha x_0(n) + \beta y_0(n)$ | $\alpha a(m) + \beta b(m)$ |
| Shifting | $x_0(n - n_0)$ | $a(m)e^{-jm\Omega_0 n_0}$ |
| Multiplication by the $M$-harmonic | $x_0(n)e^{jM\Omega_0 n}$ | $a(m - M)$ |
| Complex conjugation | $x_0^*(n)$ | $a^*(-m)$ |
| Reversal | $x_0(-n)$ | $a(-m)$ |
| Independent variable scaling $pN_0$ | $\begin{cases} x_0(n/p) & n \text{ mult. } p \\ 0 & n \text{ no mult. } p \end{cases}$ | $\frac{1}{p}a(m)$ |
| Periodic convolution | $x_0(n) * y_0(n)$ | $N_0 a(m)b(m)$ |
| Multiplication | $x_0(n)y_0(n)$ | $\sum_{k=\langle N_0 \rangle} a(k)b(k-m)$ |
| First difference | $x_0(n) - x_0(n-1)$ | $\left(1 - e^{-jm\Omega_0}\right) a(m)$ |
| Summation (finite and periodic only if $a(0) = 0$) | $\sum_{k=-\infty}^{n} x_0(k)$ | $\frac{1}{(1-e^{-jm\Omega_0})} a(m)$ |
| Inner product | $\langle x_o(n), y_o(n) \rangle = N_o \langle a(m), b(m) \rangle$ | |
| Real signals | $x_0(n)$ real | $a(m) = a^*(-m)$ |
| Even signals | $x_0(n) = x_0(-n)$ | $a(m) = a(-m)$ |
| Odd signals | $x_0(n) = -x_0(-n)$ | $a(m) = -a(-m)$ |

# Special properties: Parseval's theorem and duality

- As a consequence of the inner product property, the corresponding version of the Parseval's theorem can be derived:

$$\boxed{E_0\left[x_0(n)\right] = N_0 E_0\left[a(m)\right]}.$$

- In addition, by inspecting the analysis and synthesis equations, it can be found the following duality property:

$$\boxed{\mathbf{FS}\left(x_o(n)\right) = \frac{1}{N_o}\mathbf{FS}^{-1}\left(x_o(-m)\right)},$$

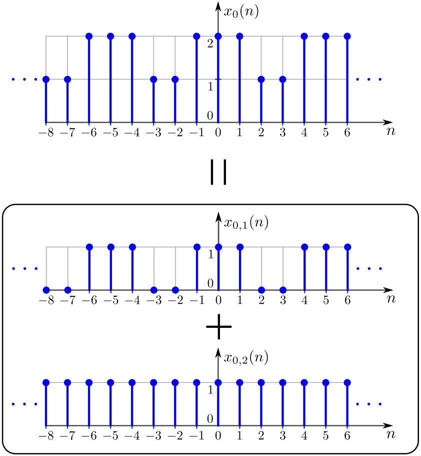
and, consequently:

$$a(n) = \frac{1}{N_o}\mathbf{DSF}^{-1}\left(x_o(-m)\right) \quad \text{o} \quad x_o(m) = N_o\mathbf{DSF}\left(a(-n)\right)$$

$$x(n) \leftrightarrow a(m) \Longleftrightarrow a(n) \leftrightarrow \frac{1}{N_0}x(-m)$$

- This property can be employ to reuse the properties and transformed pairs in both directions.

# Properties example #1




$$a(m) = \begin{cases} \frac{1}{5}\frac{\sin(3\pi k/5)}{\sin(\pi k/5)} & k \neq 0, \pm 5, \pm 10, \text{etc.} \\ \frac{8}{5} & k = 0, \pm 5, \pm 10, \text{etc.} \end{cases}$$

$$b(m) = \begin{cases} \frac{1}{5}\frac{\sin(3\pi k/5)}{\sin(\pi k/5)} & k \neq 0, \pm 5, \pm 10, \text{etc.} \\ \frac{3}{5} & k = 0, \pm 5, \pm 10, \text{etc.} \end{cases}$$

$$c(m) = \sum_k \delta(m - 5k)$$

# Properties example #2

## Exercise

Suppose we are given the following facts about the signal $x_0(n)$:

1. $x_0(n)$ is periodic, with period $N_0 = 6$
2. $\sum_{n=0}^{5} x_0(n) = 2$
3. $\sum_{n=2}^{7} (-1)^n x_0(n) = 1$
4. $x_0(n)$ has the minimum average power among the set of signals satisfying the preceding three conditions.

## Solution

$x_0(n) = \frac{1}{3} + \frac{1}{6}(-1)^n$

# Example #1

- Let us consider the following discrete-variable signal:

$$x_o(n) = e^{j\Omega_0 n} \text{ donde } \Omega_0 = \frac{2\pi}{N_0}.$$

- This signal is already express as a Fourier series and, therefore, we can directly identify the coefficients:

$$a(1) = 1$$

- And, therefore, in the range $0 \leq m \leq N_0 - 1$:

$$a(m) = \delta(m-1)$$

- If an arbitrary value of $m$ is considered, which could be out of the previous range, then we must take into account that $a(m)$ is periodic. Consequently, a general expression can be found by resorting to a periodic sum:

$$a(m) = \sum_{k=-\infty}^{\infty} \delta(m-1-kN_0)$$

# Example #2

- Let us consider another complex exponential with the same period but multiple angular frequency:

$$x_o(n) = e^{jp\Omega_0 n} \text{ where } \Omega_0 = \frac{2\pi}{N_0} \text{ and } p \in \mathbb{Z}$$

- Note that $p$ may be greater than $N_0$. Let us also assume that $p$ and $N_0$ do not have any common factor since, otherwise, the fundamental period would not be $N_0$.
- If we choose a period including the integer $p$, i.e., $N_1 \leq m \leq N_2$ so $p \in [N_1, N_2]$ and $N_2 - N_1 + 1 = N_0$, then it is straightforward to identify the only coefficient different from zero in that range:

$$a(p) = 1$$

- Therefore, in the considered range:

$$a(m) = \delta(m - p)$$

- Again, the general expression can be found by using a periodic sum:

$$\boxed{x_o(n) = e^{jp\Omega_0 n} \leftrightarrow a(m) = \sum_{k=-\infty}^{\infty} \delta(m - p - kN_0)}$$

# Example #3

- In the case of sinusoidal signals, the Fourier Series can be found by resorting to linear combinations of complex exponentials:

$$\boxed{x_o(n) = \cos\left(p\Omega_0 n\right) = \frac{1}{2}e^{jp\Omega_0 n} + \frac{1}{2}e^{-jp\Omega_0 n} \leftrightarrow \begin{array}{c} \displaystyle\frac{1}{2}\sum_{k=-\infty}^{\infty}\delta\left(m-p-kN_0\right) \\ \displaystyle+\frac{1}{2}\sum_{k=-\infty}^{\infty}\delta\left(m+p-kN_0\right) \end{array}}$$

  and

$$\boxed{x_o(n) = \sin\left(p\Omega_0 n\right) = \frac{1}{2j}e^{jp\Omega_0 n} - \frac{1}{2j}e^{-jp\Omega_0 n} \leftrightarrow \begin{array}{c} \displaystyle\frac{1}{2j}\sum_{k=-\infty}^{\infty}\delta\left(m-p-kN_0\right) \\ \displaystyle-\frac{1}{2j}\sum_{k=-\infty}^{\infty}\delta\left(m+p-kN_0\right) \end{array}}$$

  in both cases $\Omega_0 = \frac{2\pi}{N_0}$.

- Note that there are only two deltas per period.

# Example #3

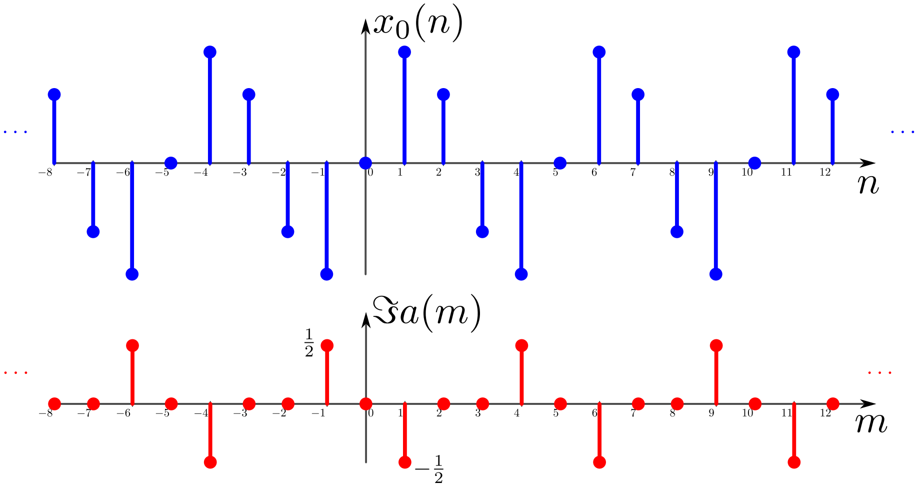



Figure: Signal $x_0(n) = \sin(2\pi/5n)$ represented in terms of FS.

# Example #3

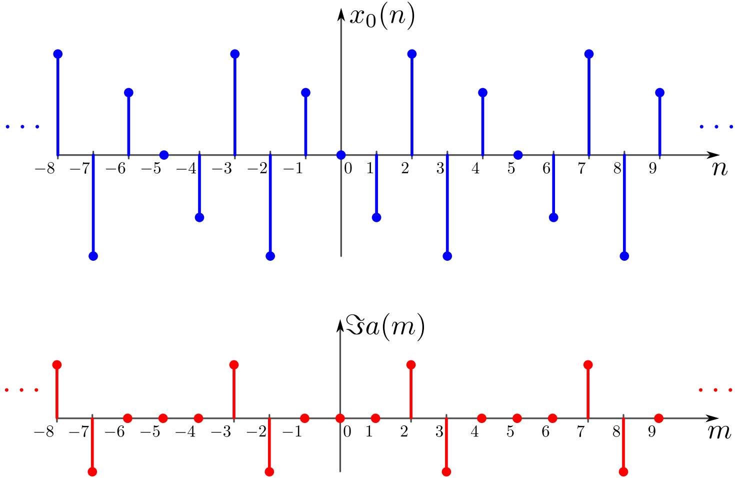



Figure: Signal $x_0(n) = \sin(2\pi \frac{3}{5} n)$ represented in terms of the FS.

# Exercise #1

## Exercise #1

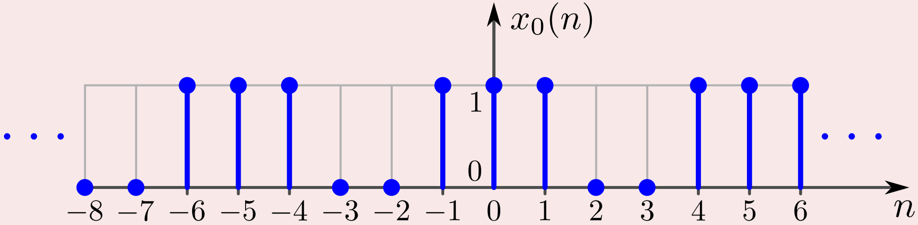
Determine the FS coefficients of the following signal

$$x_o(n) = 1 + \sin\left(\frac{2\pi}{N_0}n\right) + 3\cos\left(\frac{2\pi}{N_0}n\right) + \cos\left(\frac{4\pi}{N_0}n + \frac{\pi}{2}\right)$$

# Exercise #2

## Exercise#2

Determine the FS coefficients of the following signal



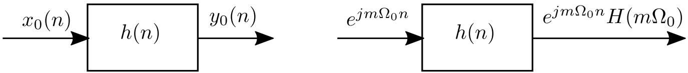

## Solution

$$a(m) = \begin{cases} \frac{1}{5}\frac{\sin(3\pi m/5)}{\sin(\pi m/5)} & m \neq 0, \pm 5, \pm 10, \text{etc.} \\ \frac{3}{5} & m = 0, \pm 5, \pm 10, \text{etc.} \end{cases}$$

# Transform pairs

| | | |
|---|---|---|
| Complex exponential | $e^{jp\Omega_o n}$ | $a(k) = 1,\ k = p \pm mN_o$ |
| Cosine | $\cos(p\Omega_o n)$ | $a(k) = 1/2,\ k = \pm p \pm mN_o$ |
| Sine | $\sin(p\Omega_o n)$ | $a(k) = \frac{1}{2j},\ k = p \pm mN_o$ <br> $a(k) = \frac{-1}{2j},\ k = -p \pm mN_o$ |
| Constant | $K$ | $a(k) = K,\ k = \pm mN_o$ |
| Train of impulses | $\delta_o(n) = \sum_{k=-\infty}^{\infty} \delta(n - kN_o)$ | $a(k) = \frac{1}{N_o},\ \forall k = p \pm mN_o$ |
| Train of pulses (width $2N+1$) | $\Pi_{o,2N+1}(n)$ | $\frac{1}{N_O} \frac{\sin\left[m\Omega_o\left(N+\frac{1}{2}\right)\right]}{\sin\left(\frac{m\Omega_o}{2}\right)}$ |

# Response of a LSI system to a complex exponential




- If the input to a LSI system is a periodic signal, then the output is given by:

$$y_0(n) = x_0(n) * h(n) = \sum_{m=\langle N_0 \rangle} a(m) \left( e^{jm\Omega_0 n} * h(n) \right).$$

- Consequently, the output signal is a linear combination of functions of the following kind:

$$e^{jm\Omega_0 n} * h(n) = \sum_{n'=\infty}^{\infty} e^{jm\Omega_0 (n-n')} h(n') = e^{j\Omega_0 mn} \sum_{n'=\infty}^{\infty} h(n') e^{-jm\Omega_0 n'}.$$

- Thus, complex exponentials are (as in other chapter), eigenfunctions of the LSI systems, and the eigenvalue $H(m\Omega_0)$ is given by:

$$H(m\Omega_0) \equiv H(\Omega)|_{\Omega=m\Omega_0} = \left[ \sum_{n'=\infty}^{\infty} h(n') e^{-j\Omega n'} \right]_{\Omega=m\Omega_0}$$

# Response of a LSI system to a periodic signal

$$x_0(n) \longrightarrow \boxed{h(n)} \xrightarrow{y_0(n)} \qquad e^{jm\Omega_0 n} \longrightarrow \boxed{h(n)} \xrightarrow{e^{jm\Omega_0 n} H(m\Omega_0)}$$

- In a similar way to the continuous-variable case, the term

$$\boxed{H(\Omega) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\Omega n}}$$

  is the (discrete-variable) Fourier transform of the system and it will be studied in the next chapter.

- The output of a LSI system to any periodic signal is also a periodic function:

$$y_0(n) = \sum_{m=\langle N_0 \rangle} b(m) e^{jm\Omega_0 n},$$

  whose FS coefficients are given by:

$$\boxed{b(m) = a(m) H(m\Omega_0)}$$

# Example of FS and LSI system

## Exercise

Find the output of a system whose impulse response is $h(n) = \alpha^n u(n)$, $-1 < \alpha < 1$ when the input is $x_0(n) = \cos\left(\frac{2\pi}{N_0} n\right)$ with $N_0 = 4$