Aperiodic Signals: the Fourier Transform

Chapter7

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Outline I

1. Introduction
   - Signal Space Algebra
   - Aperiodic signals as the limit of a periodic signal
2. Definition
   - Analysis and Synthesis Equations
   - Definition in terms of operators
3. Convergence of Fourier Transforms
   - Criteria for FT convergence
4. Properties of the FT
   - General properties
   - Parseval's theorem
5. Examples of Fourier transform
   - Finite-energy signals
   - Distributions and infinite-energy signals
   - FT of periodic signals

Outline II

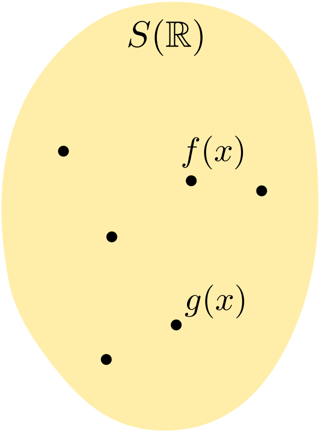
Outline

Introduction

- The FS is a tool, which enables us to represent a periodic signal as a sum of harmonically related set of complex exponentials.
- The Fourier transform (FT) is a similar tool, which is focused on aperiodic signals.
- All the previous knowledge from FS will be useful since:
  - FT can be understood as the FS when the period approaches infinity,
  - the properties are almost identical to the ones in FS,
  - the FS can be seen as particular case of the FS.
- In general, this chapter is structured analogously to the previous one (definition, convergence, properties, examples and relationship with LSI systems)

Signal Space Algebra

- It is convenient to remind the algebra for the signal space $S(\mathbb{R})$



$$\langle f(x), g(x) \rangle = \int_{-\infty}^{\infty} f(x) g^*(x) dx$$

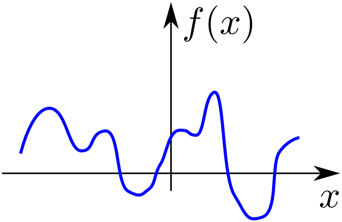
$$\|f(x)\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

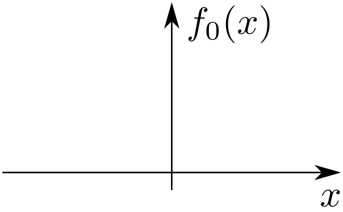
$$d^2\left(f(x), g(x)\right) = \int_{-\infty}^{\infty} |f(x) - g(x)|^2 dx$$

FS of a signal as the period approaches infinity - I

- To gain some insight into the equations yielding the FT, a conventional approach is to consider the FS of a signal as its period becomes arbitrarily large.
- For this purpose, let us consider a piece of a signal $f(x)$ so a periodic signal $f_0(x)$ is constructed by periodic summation:

$$f_0(x) = \sum_{m=-\infty}^{\infty} f(x - mX_0)\Pi\left(\frac{x - mX_0}{X_0}\right), \text{ note that } f(x) = \lim_{X_0 \to \infty} f_0(x)$$





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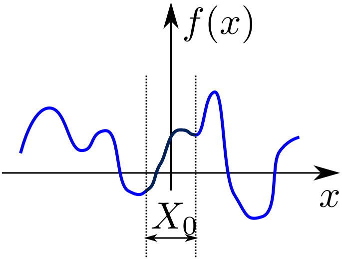

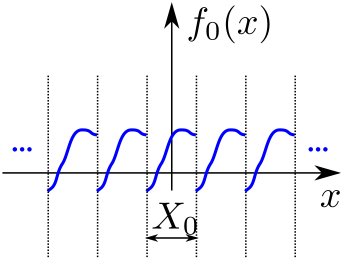

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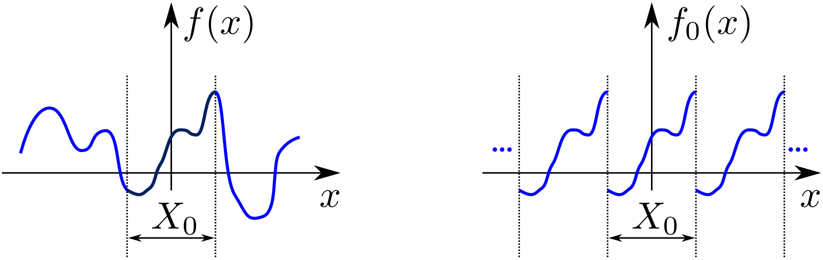

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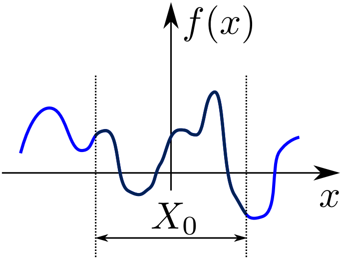
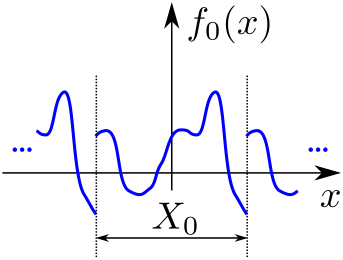




FS of a signal as the period approaches infinity - II

- As usual, the coefficients of the Fourier series are given by:

$$a(m) = \frac{1}{X_0}\int_{-\frac{X_0}{2}}^{\frac{X_0}{2}} f_0(x)e^{-jm\xi_0 x}dx$$

- If a continuous-variable function $F(\xi)$, is defined, then these coefficients can be found by evaluating this function $F(\xi)$:

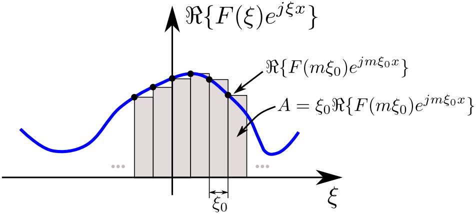
$$a(m) = \frac{1}{X_0}F(\xi = m\xi_0) \text{ where } F(\xi) = \int_{-\frac{X_0}{2}}^{\frac{X_0}{2}} f_0(x)e^{-j\xi x}dx$$

- In addition, if the FS converges:

$$f_0(x) = \sum_{m=-\infty}^{\infty} a(m)e^{jm\xi_0 x} = \sum_{m=-\infty}^{\infty} \frac{1}{X_0}F(m\xi_0)e^{jm\xi_0 x} =$$

$$\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \xi_0 F(m\xi_0)e^{jm\xi_0 x} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \xi_0 \left[F(\xi)e^{j\xi x}\right]_{\xi=m\xi_0}$$

FS of a signal as the period approaches infinity - III



As X_0 approaches infinity, the sum becomes a Riemann integral (for both, the real and the imaginary parts).

- Considering the limit as $X_0 \to \infty$, then we have:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\xi)e^{j\xi x}d\xi \quad \text{where} \quad F(\xi) = \int_{-\infty}^{\infty} f(x)e^{-j\xi x}dx$$

Representation of an aperiodic signal

- The previous study reveals that an aperiodic signal can be (generally) expanded as a generalized linear combination of complex exponentials.

$$\boxed{f(x) = \frac{1}{2\pi}\mathbf{LC}\left\{F(\xi)\varphi(x;\xi)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\xi)e^{j\xi x}d\xi}$$

- Thus, it suggests that exponentials with imaginary argument

$$\boxed{\{\varphi(x;\xi)\} = e^{j\xi x}}$$

  can be useful to generate (most of the) signals in $S(\mathbb{R})$. In other words, it is a spanning set and, moreover, their elements are orthogonal (see next slide) so it is a basis.
- In addition, the coefficients of the linear combination can be also identified as an inner product with the elements of the basis:

$$\boxed{F(\xi) = \langle f(x), \varphi(x;\xi)\rangle = \int_{-\infty}^{\infty} f(x)e^{-j\xi x}dx}$$

Orthogonality of the basis

$$\boxed{f(x)=\frac{1}{2\pi}\mathbf{LC}\left\{F(\xi)\varphi(x;\xi)\right\}} \qquad \boxed{F(\xi)=\langle f(x),\varphi(x;\xi)\rangle} \qquad \boxed{\{\varphi(x;\xi)\}=e^{j\xi x}}$$

- Note that the previous expressions are identical to those ones for representing a signal in terms of an uncountable orthogonal basis.
  - Nevertheless, it will be shown how this orthogonality is in the sense of the distributions.
- One of the main advantages of an orthogonal basis is that the coefficients of the linear combinations, $F(\xi)$, can be found by means of a simple inner product:

$$F(\xi)=\langle f(x),\varphi(x;\xi)\rangle$$

if the linear combination converges, then:

$$F(\xi)=\langle f(x),\varphi(x;\xi)\rangle=\frac{1}{2\pi}\left\langle \mathbf{CL}\left\{F(\xi')\varphi(x;\xi')\right\},\varphi(x;\xi)\right\rangle$$
$$=\frac{1}{2\pi}\mathbf{CL}\left\{F(\xi')\left\langle\varphi(x;\xi'),\varphi(x;\xi)\right\rangle\right\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\xi')\left\langle\varphi(x;\xi'),\varphi(x;\xi)\right\rangle d\xi'$$

if the left and right hand side are always equal, then

$$\boxed{\left\langle\varphi(x;\xi'),\varphi(x;\xi)\right\rangle=2\pi\delta\left(\xi-\xi'\right)}$$

Outline

The Fourier Transform

Analysis and Synthesis Equations

Taking into account the previous definitions, the analysis and synthesis equations can be defined:

$$\boxed{F(\xi) = \int_{-\infty}^{\infty} f(x) e^{-j\xi x} dx} \text{ (analysis eq.)}$$

$$\boxed{f(x) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\xi) e^{j\xi x} d\xi} \text{ (synthesis eq.)}$$

These equations define the Fourier Transform (FT) for continuous-variable signal.

The symbol $\approx$ is used because, in a similar fashion to the FS, it is not always possible to retrieve the original signal.

Definition in terms of operators

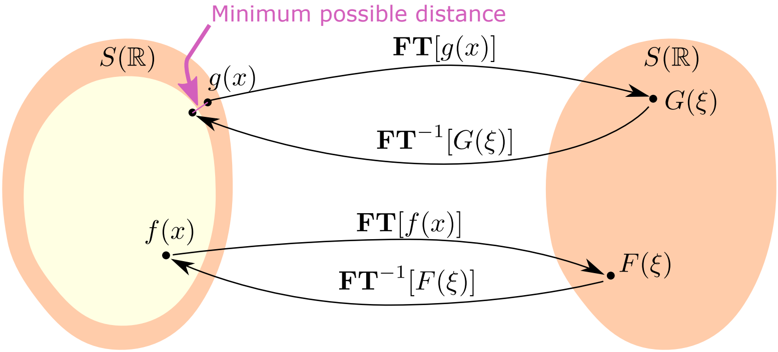
- The Fourier transform can be also defined in terms of a (non-invertible) operator between two signal spaces, which are the same one in contrast to the FS case:

$$\begin{aligned}\mathbf{FT} : S(\mathbb{R}) &\to S(\mathbb{R}) \\ f(x) &\mapsto F(\xi) = \int_{-\infty}^{\infty} f(x) e^{-j\xi x} dx\end{aligned}$$

- In a similar fashion, the inverse FT

$$\begin{aligned}\mathbf{FT}^{-1} : S(\mathbb{R}) &\to S(\mathbb{R}) \\ F(\xi) &\mapsto \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\xi) e^{j\xi x} d\xi \approx f(x)\end{aligned}$$

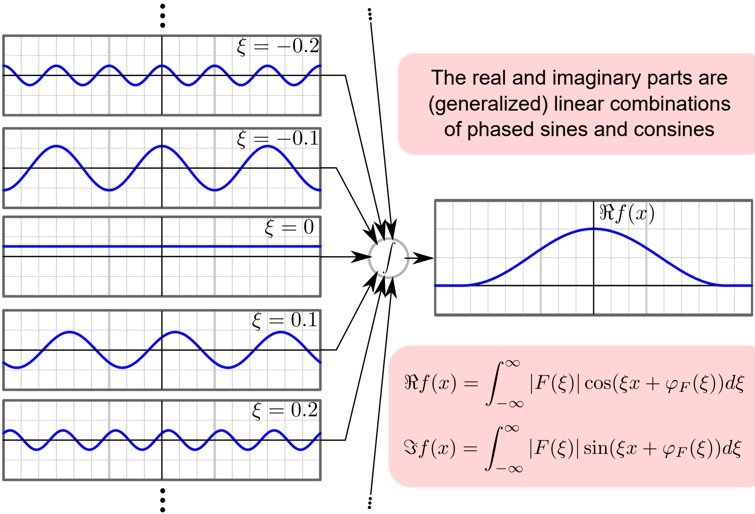
Definition in terms of operators



$$f(x) = \mathbf{FT}^{-1}[F(\xi)] \qquad g(x) \neq \mathbf{FT}^{-1}[G(\xi)]$$

- Despite converge is not guaranteed, it is possible to show that the FT exists and converges for most of the signals of interest in engineering.
- In addition, the FT definition can be extended to work with periodic functions and distributions.

Example: Gaussian function



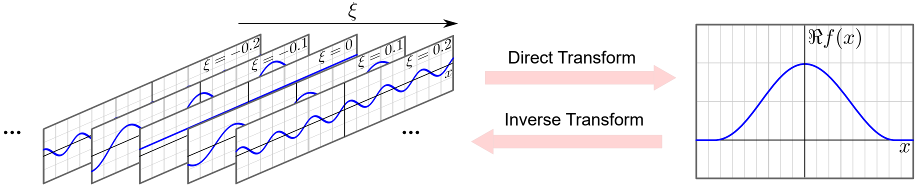
The real and imaginary parts are (generalized) linear combinations of phased sines and consines

$$\Re f(x) = \int_{-\infty}^{\infty} |F(\xi)| \cos(\xi x + \varphi_F(\xi)) d\xi$$

$$\Im f(x) = \int_{-\infty}^{\infty} |F(\xi)| \sin(\xi x + \varphi_F(\xi)) d\xi$$

Example: Gaussian function

The Fourier transform enables us to decomposed a signal into sinuoidal signals.
The amplitude and phase of the sinusoids yield the transformed signal.



$$\Re f(x) = \int_{-\infty}^{\infty} |F(\xi)| \cos(\xi x + \varphi_F(\xi)) d\xi \qquad \Im f(x) = \int_{-\infty}^{\infty} |F(\xi)| \sin(\xi x + \varphi_F(\xi)) d\xi$$

Outline

Criteria for FT convergence

- In a similar fashion to the FS case, the existence of the coefficients $F(\xi)$, which are given by an integral over $\mathbb{R}$, is not guaranteed.
- Moreover, even if the FS $F(\xi)$ exits, it is not guaranteed that the inverse transform will converge to retrieve the original signal $f(x)$.
- Despite there is a large number of convergence criteria, we will just focus on two criteria, which are very similar to the ones already used in the FS.
    - It is intuitive to understand that the convergence criteria for the FT will be identical to the ones in FS since an aperiodic signal can be constructed from a periodic signal when its period approaches infinity.
- In practical contexts, real-world signals can be represented in terms of FT.

Concerns about the convergence of FT

- It is worthwhile mentioning that, in contrast to the FS, even if there is a singularity in $F(\xi)$, it does not implies that the inverse Fourier transform does not exist.
- The reason is that an integral can convergence even if there are singularities at isolated points.

Criteria for FT convergence

Convergence criteria

The FT of a signal $f(x)$ exists and convergences if it meets any of the following two conditions:

- $f(x) \in L^2$, i.e., it is square integrable (finite energy),
- $f(x) \in L^1$, i.e., it is absolutely integrable, and, in addition, it has a finite number of (a) maxima and minima, and (b) discontinuities, in any finite-length interval. Besides, all the discontinuities must be finite.

It is important to remark that these conditions are sufficient (but not necessary).

- Each of these criteria yields a different kind of convergence.

Convergence of the FT - Signals in L^2

- The first convergence criterion corresponds to $f(x) \in L^2$:

$$||f(x)||^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx = E\left[f(x)\right] < \infty$$

- The FT of every finite-energy signal converges in the sense of the 2-norm (norm induced by the inner product):

$$d\left(f(x), \mathbf{TF}^{-1}\left(F\left(\xi\right)\right)\right) = \left\| f(x) - \mathbf{TF}^{-1}\left(F\left(\xi\right)\right)\right\| = 0$$

- This fact is very relevant because the energy of signals generated in a laboratory is always finite.

Convergence of the FT -Dirichlet conditions

- As in the FS, the second convergence criterion is also referred to as Dirichlet conditions:
  1. $f(x) \in L^1$:
  $$\int_{-\infty}^{\infty} |f(x)| dx < \infty$$
  2. In any finite interval, there are only a finite number of discontinuities. Moreover, each of these discontinuities is finite.
  3. In any finite interval, there are only a finite number of maxima and minima.
- Signals not satisfying the two last conditions are generally pathological and they do not arise in practical contexts.
- If Dirichlet conditions are met, the FT converges to $f(x)$ everywhere except at the points of discontinuity.
  - In the discontinuities, the FT converges to the average value at the point of discontinuity.

Convergence of the FT - exercise

Exercise

Evaluate if the function $f(x) = \frac{1}{\sqrt{|x|}}$, meets any of the convergence criteria.

Outline

General properties of the FT

Linearity	$\alpha f(x)+\beta g(x)$	$\alpha F(\xi)+\beta G(\xi)$
Shifting	$f(x-x_o)$	$F(\xi)e^{-j\xi x_0}$
Multiplication by a complex exponential	$f(x)e^{j\xi_o x}$	$F(\xi-\xi_0)$
Complex conjugate	$f^*(x)$	$F^*(-\xi)$
Reversal	$f(-x)$	$F(-\xi)$
Scaling	$f(ax)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\xi}{a}\right)$
Convolution	$f(x)*g(x)$	$F(\xi)G(\xi)$
Multiplication	$f(x)g(x)$	$\frac{1}{2\pi}F(\xi)*G(\xi)$
First derivative	$\frac{d}{dx}f(x)$	$j\xi F(\xi)$
High order derivatives	$\frac{d^n}{dx^n}f(x)$	$(j\xi)^n F(\xi)$
Multiplication by x^n, $n>0$	$x^n f(x)$	$j^n\frac{d^n F(\xi)}{d\xi^n}$
Integral	$\int_{-\infty}^{x} f(y)dy$	$\frac{1}{j\xi}F(\xi)+\pi F(0)\delta(\xi)$
Duality	$F(x)$	$2\pi f(-\xi)$
Inner product	$\langle f(x),g(x)\rangle=\frac{1}{2\pi}\langle F(\xi),G(\xi)\rangle$	
Real signals	$f(x)$ real	$F(\xi)=F^*(-\xi)$
Even signals	$f(x)=f(-x)$	$F(\xi)=F(-\xi)$
Odd signals	$f(x)=-f(-x)$	$F(\xi)=-F(-\xi)$

($F(\xi)$ and $G(\xi)$ represent the FT of $f(x)$ and $g(x)$, respectively)

Parseval's (or Plancherel's) Theorem

- Taking into account the inner product property, it is trivial to prove the following theorem.

Parseval's (or Plancherel's) Theorem

The energy of a signal $f(x)$ equals the energy corresponding the signal of its Fourier transform divided by 2π.

$$\boxed{\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\xi)|^2 d\xi}$$

$$\boxed{E\left[f(x)\right] = \frac{1}{2\pi} E\left[F(\xi)\right]}$$

Example of FT properties

Exercise

1. Use the analysis equation to calculate $\textbf{FT}\left[e^{-ax}u(x)\right]$ for $\Re a > 1$.
2. Considering the properties of the FT and the previous result, determine $\textbf{FT}\left[e^{-a|x|}\right]$.
3. Considering the properties of the FT and the previous result, determine $\textbf{FT}\left[\frac{2}{1+x^2}\right]$.
4. Taking into account the properties of the FT and the previous result, calculate the result of the following integral $\int_{-\infty}^{\infty} \frac{2}{1+x^2} dx$.

Exercise

Considering the result from the first section of the exercise, and taking into account that

$$\frac{d}{d\xi}\left(\frac{1}{a+j\xi}\right) = \frac{1}{(a+j\xi)^2}$$

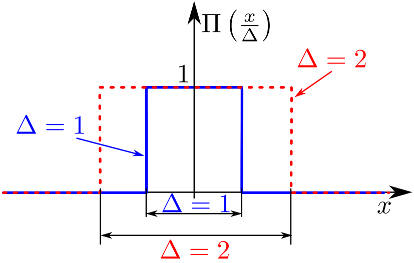
find the inverse FT of:

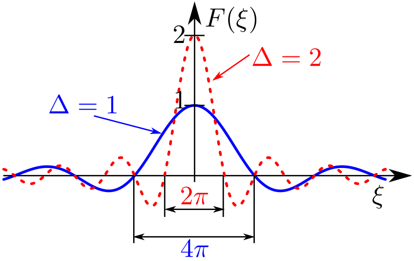
$$\frac{1}{(a+j\xi)^2}$$

Outline

Unit pulse

$$\boxed{f(x) = \Pi\left(\frac{x}{\Delta}\right) \longleftrightarrow F(\xi) = \Delta \mathrm{sinc}\left(\frac{\Delta}{2}\xi\right)}$$





- Note that:
  - the narrower the pulse, the wider the main lobe in the spectral domain.
  - the properties for real and even signals are met.

Unit pulse

Proof

The proof is straightforward by resorting to the definition of the FT:

$$\int_{-\infty}^{\infty} f(x)e^{-j\xi x}dx = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^{-j\xi x}dx = \frac{1}{-j\xi}\, e^{-j\xi x}\Big|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} =$$

$$\frac{e^{j\frac{\Delta}{2}\xi} - e^{-j\frac{\Delta}{2}\xi}}{j\xi} = 2\frac{\sin\left(\frac{\Delta}{2}\xi\right)}{\xi} = 2\frac{\Delta}{2}\frac{\sin\left(\frac{\Delta}{2}\xi\right)}{\xi\frac{\Delta}{2}} = \Delta\mathrm{sinc}\left(\frac{\Delta}{2}\xi\right)$$

Sinc function

- Taking into account the previous result for a unit step, and considering the duality property, then it is also simple to find the FT of a sinc signal:

$$\Pi\left(\frac{x}{\Delta}\right) \longleftrightarrow \Delta \mathrm{sinc}\left(\frac{\Delta}{2}\xi\right)$$

$$\Delta \mathrm{sinc}\left(\frac{\Delta}{2}x\right) \longleftrightarrow 2\pi\Pi\left(\frac{-\xi}{\Delta}\right) = 2\pi\Pi\left(\frac{\xi}{\Delta}\right)$$

$$\mathrm{sinc}\left(\frac{\Delta}{2}x\right) \longleftrightarrow \frac{2\pi}{\Delta}\Pi\left(\frac{\xi}{\Delta}\right)$$

$$\boxed{\frac{\Delta}{2\pi}\mathrm{sinc}\left(\frac{\Delta}{2}x\right) \longleftrightarrow \Pi\left(\frac{\xi}{\Delta}\right)}$$

Unit step

Exercise

Show that

$$\mathrm{sinc}\xi_1 x * \mathrm{sinc}\xi_2 x = \mathrm{sinc}\xi_3 x$$

and determine the value of ξ_3.

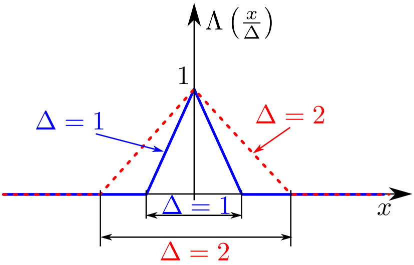
Exercise

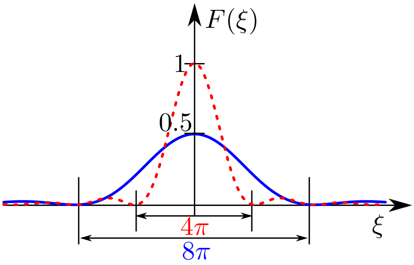
Compute the following inner product

$$\left\langle e^{j\xi_0 x}\mathrm{sinc}\xi_0 x, e^{-j\xi_0 x}\mathrm{sinc}\xi_0 x \right\rangle$$

Triangle function

$$\boxed{f(x) = \Lambda\left(\frac{x}{\Delta}\right) \longleftrightarrow F(\xi) = \frac{\Delta}{2}\mathrm{sinc}^2\left(\frac{\Delta}{4}\xi\right)}$$





- Note that:
  - the narrower the triangle, the wider the main lobe in the spectral domain.
  - the properties for real and even signals are met.

Triangle function

Proof

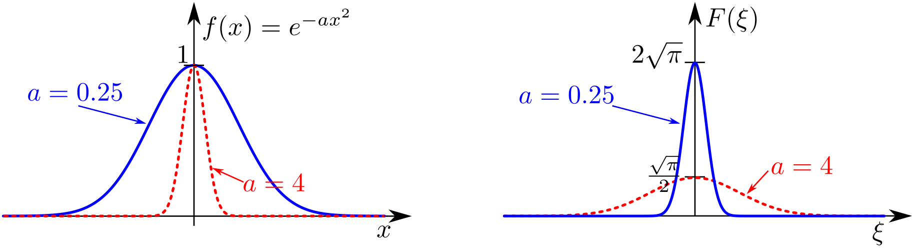
The proof is easy by taking into account the convolution property:

$$\Lambda\left(\frac{x}{\Delta}\right) = \frac{2}{\Delta}\Pi\left(\frac{x}{\Delta/2}\right) * \Pi\left(\frac{x}{\Delta/2}\right)$$

$$\Lambda\left(\frac{x}{\Delta}\right) \longleftrightarrow \frac{2}{\Delta}\left(\frac{\Delta}{2}\mathrm{sinc}\left(\frac{\Delta}{4}\xi\right)\right)^2 = \frac{\Delta}{2}\mathrm{sinc}^2\left(\frac{\Delta}{4}\xi\right)$$

Gaussian function

$$\boxed{f(x) = e^{-ax^2} \longleftrightarrow F(\xi) = \sqrt{\frac{\pi}{a}} e^{-\frac{\xi^2}{4a}}} \text{ where } \Re a > 0$$



- Note that:
  - the FT of the Gaussian function is also a Gaussian function,
  - the narrower (i.e., the faster is changes) the signal $f(x)$, the wider its FT $F(\xi)$ (i.e., higher frequencies are required),
  - the properties for real and even signals are met.

Gaussian function

Proof

The proof can be easily accomplished by considering the following known result

$$\int_{-\infty}^{\infty} e^{-\left(ax^2+bx+c\right)}dx = \sqrt{\frac{\pi}{a}}e^{\frac{\left(b^2-4ac\right)}{4a}}, \text{ where } \Re a > 0$$

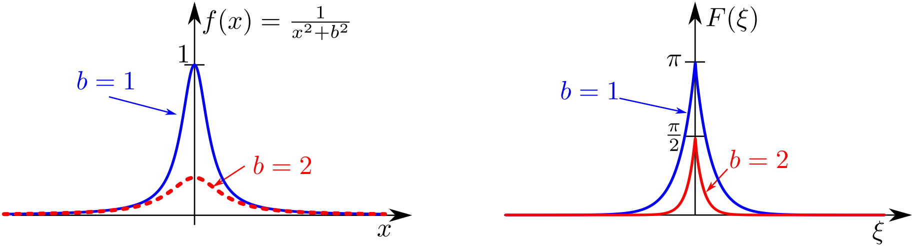
by taking $b = j\xi$ and $c = 0$.
The previous relation can be proved by considering the well-known result (particularly useful in probability theory) for

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$$

by completing the square and a change of variable. Nevertheless, the rigorous proof requires basic knowledge of contour integrals (i.e., complex-variable integrals) to generalize the result for complex numbers.

Pseudo-Gaussian function

$$\boxed{f(x) = \frac{1}{x^2+b^2} \longleftrightarrow F(\xi) = \frac{\pi}{b} e^{-b|\xi|}}$$



- Note that:
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Pseudo-Gaussian function

Proof

The proof is easy from the following result

$$\int_0^\infty \frac{\cos\beta x}{x^2+b^2}dx = \frac{\pi}{2b}e^{-b|\beta|}$$

which can be also found by contour integration.
Thus, the proof is as follows:

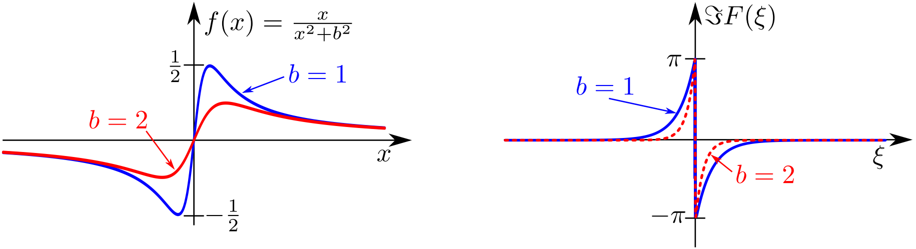
$$\int_{-\infty}^\infty \frac{1}{x^2+b^2}e^{-j\xi x}dx = \int_{-\infty}^\infty \frac{\cos\xi x}{x^2+b^2}dx + j\int_{-\infty}^\infty \frac{\sin\xi x}{x^2+b^2}dx$$

since the two integrands are even and odd, respectively, then:

$$\int_{-\infty}^\infty \frac{1}{x^2+b^2}e^{-j\xi x}dx = 2\int_0^\infty \frac{\cos\xi x}{x^2+b^2}dx = \frac{\pi}{b}e^{-b|\xi|}$$

Pseudo-Gaussian function multiplied by x

$$\boxed{f(x) = \frac{x}{x^2+b^2} \longleftrightarrow F(\xi) = -j\pi \operatorname{sgn}(\xi)\, e^{-b|\xi|}}$$



- Note that que:
  - the narrower (i.e., the faster is changes) the signal $f(x)$, the wider its FT $F(\xi)$ (i.e., higher frequencies are required),
  - the properties for real and odd signals are met.

Pseudo-Gaussian function multiplied by x

Proof

The proof can be constructed from the FT of the pseudo-Gaussian function and the property regarding the multiplication by x

$$xf(x) = \frac{x}{x^2+b^2} \longleftrightarrow -j\frac{d}{d\xi}\left(\frac{\pi}{b}e^{-b|\xi|}\right)$$

Thus, it yields:

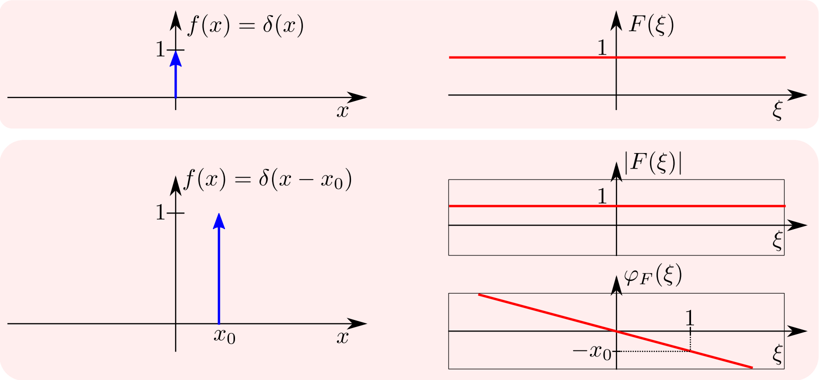
$$j\frac{d}{d\xi}\left(\frac{\pi}{b}e^{-b|\xi|}\right) = -j\pi\frac{d|\xi|}{d\xi}e^{-b|\xi|} = -j\pi\mathrm{sgn}\left(\xi\right)e^{-b|\xi|}$$

Fourier transforms of distributions

- The concept of Fourier transform can be generalized to consider distributions (see appendix).
  - This generalized transform is defined so most of the FT properties are met (duality, derivatives, shifting, convolutions and parity)
- It is possible to show that in the case of impulse distributions ($\delta(x)$, $\delta'(x)$, etc.) and associated distributions (e.g., the primitive $\Gamma(x)$), regular operations lead to the same result.
- Moreover, distributions will enable us to define the FT of significant number of useful functions, which cannot be defined in terms of regular functions.
  - A good example are the periodic signals, whose FT is defined in terms of distributions.

Dirac Delta

$$\boxed{\delta(x) \longleftrightarrow 1} \qquad \boxed{\delta(x-x_0) \longleftrightarrow e^{-j\xi x_0}}$$



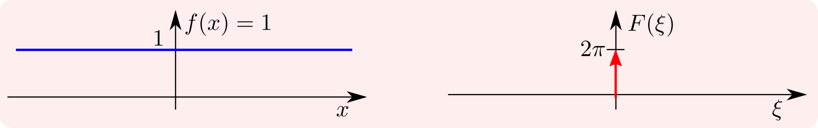
Proof

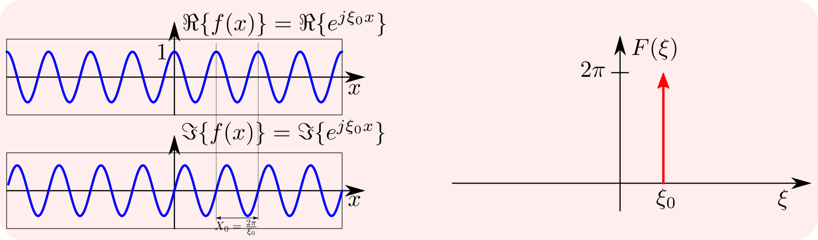
It is trivial by considering the definition of Dirac Delta $\int_{-\infty}^{\infty} \delta(x-x_0)e^{-j\xi x}dx = e^{-j\xi x_0}$

Dirac Delta

- In addition, considering the duality property:

$$\boxed{1 \longleftrightarrow 2\pi\delta(\xi)} \qquad \boxed{e^{j\xi_0 x} \longleftrightarrow 2\pi\delta(\xi-\xi_0)}$$

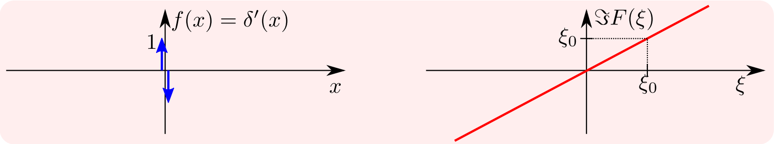


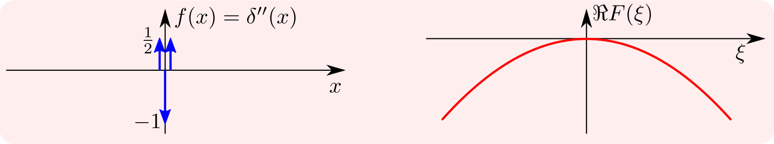


Derivatives of the Dirac Delta

- In a similar way to the previous case:

$$\boxed{\delta^{(n)}(x) \longleftrightarrow (j\xi)^n} \qquad \boxed{\delta^{(n)}(x - x_0) \longleftrightarrow (j\xi)^n\, e^{-j\xi x_0}}$$





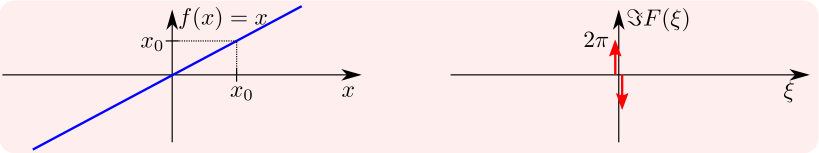
Proof

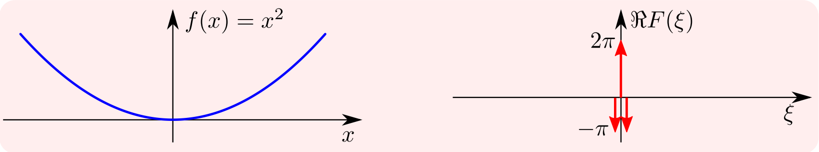
$$\int_{-\infty}^{\infty} \delta^{(n)}(x - x_0) e^{-j\xi x} dx = (-1)^n \left. \frac{d^n \left(e^{-j\xi x}\right)}{dx^n} \right|_{x = x_0} = (-1)^n (-j\xi)^n e^{-j\xi x_0} = (j\xi)^n e^{-j\xi x_0}$$

Derivatives of the Dirac Delta

- Taking into account the duality property and the previous result:

$$\boxed{x^n \longleftrightarrow 2\pi j^n \delta^{(n)}(\xi)}$$





Proof

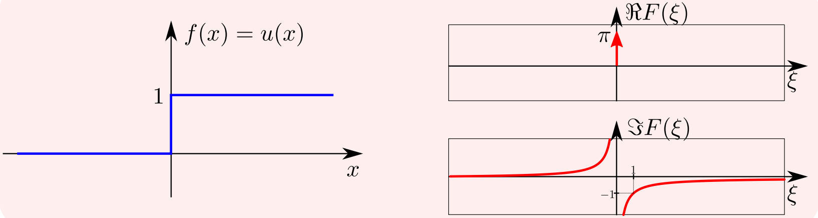
$$(jx)^n \longleftrightarrow 2\pi\delta^{(n)}(-\xi) = (-1)^n 2\pi\delta^{(n)}(\xi)$$

$$x^n \longleftrightarrow 2\pi \frac{1}{j^n}(-1)^n \delta^{(n)}(\xi) = 2\pi j^n \delta^{(n)}(\xi)$$

Unit step function

- The FT of a unit step can be also defined in terms of the distributions:

$$\boxed{u(x) \longleftrightarrow \pi\delta(\xi) + \frac{1}{j\xi}}$$



Exercise

Prove the integration property by resorting to the FT of the unit step.

$$\int_{-\infty}^{x} f(y)dy \leftrightarrow \frac{1}{j\xi}F(\xi) + \pi F(0)\delta(\xi)$$

Unit step function

Proof

The (casual) proof is based on the property of the derivative of the FT and the fact of $u'(x) = \delta(x)$

$$1 = \mathbf{FT}\left(\delta(x)\right) = \mathbf{FT}\left(u'(x)\right) = j\xi\mathbf{FT}\left(u(x)\right)$$

isolating the FT from the previous identity, and considering the results from chapter on distributions, we have:

$$\mathbf{FT}\left(u(x)\right) = \frac{1}{j\xi} + C\delta(\xi)$$

wherein C is a constant to be determined. For this purpose, let us consider the following result

$$\mathbf{FT}\left(\text{sgn}(x)\right) = \mathbf{FT}\left(2u(x) - 1\right) = 2\mathbf{FT}\left(u(x)\right) - 2\pi\delta\left(\xi\right) = \frac{2}{j\xi} + \left(2C - 2\pi\right)\delta(\xi)$$

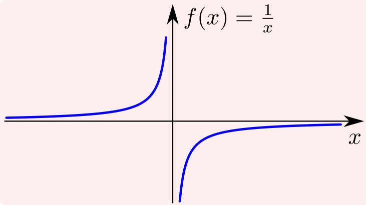
since the $\mathbf{FT}\left(\text{sgn}(x)\right)$ must be an odd function, the term $\delta(\xi)$ must be canceled out because it corresponds to an even distribution and, therefore:

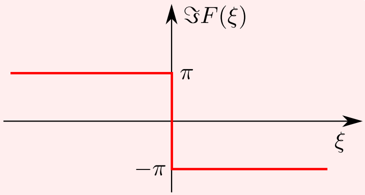
$$2C - 2\pi = 0 \Leftrightarrow C = \pi$$

Unit step function

- Considering the duality property:

$$\boxed{\frac{1}{x} \longleftrightarrow -\pi j \operatorname{sgn}(\xi)}$$





Proof

$$u(x) \longleftrightarrow \pi\delta(\xi) + \frac{1}{j\xi} \quad \Leftrightarrow \quad \pi\delta(x) + \frac{1}{jx} \longleftrightarrow 2\pi u(-\xi) \quad \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{jx} \longleftrightarrow 2\pi u(-\xi) - \mathbf{FT}(\pi\delta(x)) = 2\pi u(-\xi) - \pi \quad \Leftrightarrow \quad \frac{1}{x} \longleftrightarrow 2\pi j\left(u(-\xi) - \frac{1}{2}\right) = \pi j \operatorname{sgn}(-\xi)$$

FT of periodic signals

- Distributions are also useful to calculate the FT of periodic functions.
- For this purpose, it is enough to consider that if a function can represented by a FS, then

$$f_0(x) = \sum_{m=-\infty}^{\infty} a(m) e^{jm\xi_0 x}$$

- Taking into account that **FT** $\left[e^{jm\xi_0 x}\right] = 2\pi\delta(\xi - m\xi_0)$:

$$f_0(x) = \sum_{m=-\infty}^{\infty} a(m) e^{jm\xi_0 x} \longleftrightarrow F_0(\xi) = 2\pi \sum_{m=-\infty}^{\infty} a(m)\delta(\xi - m\xi_0)$$

- As it was shown in the introduction, the FS coefficients can be calculated from the FT of the signal defined over a single period:

$$a(m) = \frac{1}{X_0}\int_{-\frac{X_0}{2}}^{\frac{X_0}{2}} f_0(x) e^{-jm\xi_0 x} dx = \frac{1}{X_0} F(m\xi_0) \quad \text{where } F(\xi) = \int_{-\frac{X_0}{2}}^{\frac{X_0}{2}} f_0(x) e^{-j\xi x} dx$$

- It is possible to choose another interval of integration, e.g., $[x_0, X_0 + x_0]$, and, although $F(\xi)$ will be generally different, the result will be identical at the points $\xi = m\xi_0$.

FT of periodic signals

FT of periodic signals

The FT of a periodic signal with period X_0, is composed of a train of Dirac Deltas with a spacing of $\xi_0 = \frac{2\pi}{X_0}$, whose weights are proportional to the coefficients of the FS.

$$\boxed{f_0(x) = \sum_{m=-\infty}^{\infty} a(m) e^{jm\xi_0 x} \longleftrightarrow F_0(\xi) = 2\pi \sum_{m=-\infty}^{\infty} a(m)\delta(\xi - m\xi_0)}$$

In addition, the FS coefficients can be calculated from the FT of the signal <u>over a single period</u> as:

$$\boxed{a(m) = \frac{1}{X_0} F(m\xi_0)}$$

$$\boxed{F(\xi) = \int_{x_0}^{x_0+X_0} f_0(x) e^{-j\xi x} dx, \forall x_0}$$

FT of periodic signals

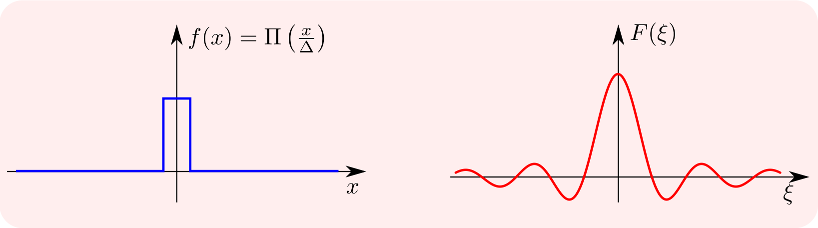
Exercise

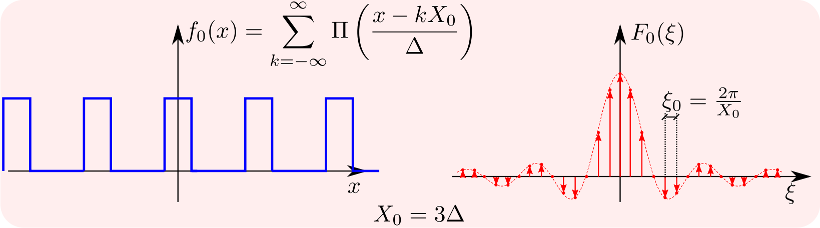
Determine the FT of a pulse train $f_0(x) = \sum_{m=-\infty}^{\infty} \Pi\left(\frac{x-mX_0}{\Delta}\right)$ from the FT of a pulse $\Pi\left(\frac{x}{\Delta}\right)$.

Solution

$$\boxed{\Pi_{0,\Delta}(x) = \sum_{m=-\infty}^{\infty} \Pi\left(\frac{x-mX_0}{\Delta}\right) \longleftrightarrow \frac{2\pi\Delta}{X_0} \sum_{m=-\infty}^{\infty} \mathrm{sinc}\left(m\pi\frac{\Delta}{X_0}\right)\delta\left(\xi - m\xi_0\right)}$$

FT of periodic signals





Some periodic signals of interest

- Taking into account the previous result, and by means of a simple decomposition into exponentials, it is trivial to calculate the FT of sine and cosine functions:

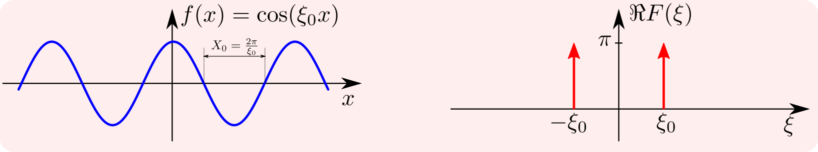
$$\boxed{\cos\left(\xi_0 x\right) \longleftrightarrow \pi\delta\left(\xi-\xi_0\right)+\pi\delta\left(\xi+\xi_0\right)}$$

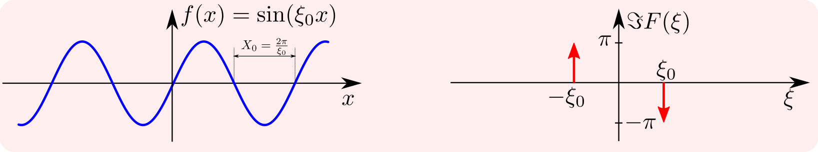
$$\boxed{\sin\left(\xi_0 x\right) \longleftrightarrow \frac{\pi}{j}\delta\left(\xi-\xi_0\right)-\frac{\pi}{j}\delta\left(\xi+\xi_0\right)}$$

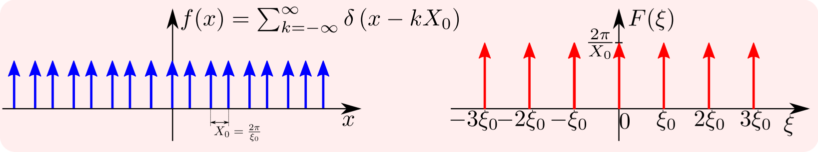
- In addition, it is simple to show the following result by means of the Poisson summation formula:

$$\boxed{\sum_{k=-\infty}^{\infty}\delta\left(x-kX_0\right) \longleftrightarrow \frac{2\pi}{X_0}\sum_{m=-\infty}^{\infty}\delta\left(\xi-m\xi_0\right),\ \text{where } \xi_0=\frac{2\pi}{X_0}}$$

Some periodic signals of interest







Outline

Response of a LSI system to a complex exponential

- As shown in the previous chapter, complex exponentials are eigenfunctions of the linear and shift-invariant systems:

$$\mathbf{F}\left[\varphi(x;\xi)\right] = \varphi(x;\xi) * h(x) = \int_{-\infty}^{\infty} h(x')e^{j\xi(x-x')}dx' = e^{j\xi x}\int_{-\infty}^{\infty} h(x')e^{-j\xi x'}$$

$$\xrightarrow{e^{j\xi x}} \boxed{h(x)} \xrightarrow{e^{j\xi x}H(\xi)} \qquad \boxed{H(\xi) = \int_{-\infty}^{\infty} h(x)e^{-j\xi x}dx}$$

- The function

$$\boxed{H(\xi) = \mathbf{FT}\left[h(x)\right]}$$

is referred to as transfer function of the system.

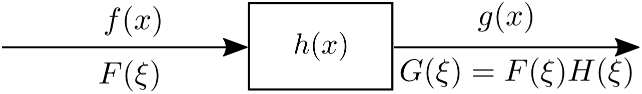
Response of a LSI system to an arbitrary signal

- Consequently, the response of a LSI to a signal $f(x)$ is:

$$g(x) = \mathbf{F}\left[f(x)\right] = \int_{-\infty}^{\infty} F(\xi)\mathbf{F}\left[\varphi(x;\xi)\right] d\xi = \int_{-\infty}^{\infty} F(\xi)H(\xi)e^{j\xi x} d\xi$$

- Thus, the relationship between the input and output spectra is given by:

$$\boxed{G(\xi) = F(\xi)H(\xi)}$$

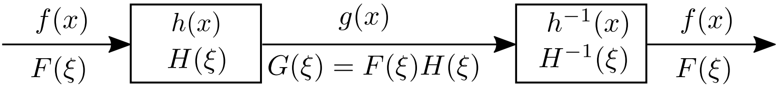


Inverse system of a LSI

- The transfer function of the inverse system is given by:

$$\mathbf{FT}\left[h^{-1}(x)\right] = \frac{1}{H(\xi)}$$

- Thus, it is necessary that $H(\xi) \neq 0, \forall \xi$ so that the system can be invertible.



Outline

Energy spectral density

- According to the Parseval's theorem:

$$E\left[f\left(x\right)\right] = \int_{-\infty}^{\infty} \left|f\left(x\right)\right|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|F\left(\xi\right)\right|^2 d\xi \quad [J]$$

- Therefore, the term $G_f\left(\xi\right) = \left|F\left(\xi\right)\right|^2$ describes the density of the energy over the spectral domain and we can assume that the corresponding units are $[J/(rad/s)]$ or $[J/(rad/m)]$ for the time- and space-domains, respectively.
- The term $\boxed{G_f\left(\xi\right) = \left|F\left(\xi\right)\right|^2}$ is usually referred to as energy spectral density.

Wiener-Kinchine theorem

Wiener-Kinchine theorem

The energy power density is related to the autocorrelation by means of the Fourier transform:

$$\boxed{R_f(x) \leftrightarrow G_f(\xi)}$$

The proof is straightforward if the FT properties are applied to the definition of autocorrelation $R_f(x) = f(x) * f^*(-x)$.
Similar relationship are also available for the crosscorrelation:

$$\begin{aligned} R_{fg}(x) &\leftrightarrow& G_{fg}(\xi) = F(\xi)G^*(\xi) \\ R_{gf}(x) &\leftrightarrow& G_{gf}(\xi) = G(\xi)F^*(\xi) \end{aligned}$$

Example of the Wiener-Kinchine theorem

- Example of the Wiener-Kinchine theorem:

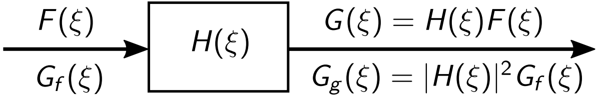
$$
\begin{tikzcd}
f(x) \arrow[rrr, leftrightarrow, "\langle\cdot,\cdot\rangle"] \arrow[d, leftrightarrow, "\mathrm{FT}"'] & & & R_f(x) \arrow[d, leftrightarrow, "\mathrm{FT}"] \\
F(\xi) \arrow[rrr, leftrightarrow, "|\cdot|^2"'] & & & G_f(\xi)
\end{tikzcd}
$$

$$
\begin{tikzcd}
f(x) = \Pi(\frac{x}{\Delta}) \arrow[rrr, leftrightarrow, "\langle\cdot,\cdot\rangle"] \arrow[d, leftrightarrow, "\mathrm{FT}"'] & & & R_f(x) = \Delta\Lambda\left(\frac{x}{2\Delta}\right) \arrow[d, leftrightarrow, "\mathrm{FT}"] \\
F(\xi) = \Delta\mathrm{sinc}(\frac{\Delta}{2}\xi) \arrow[rrr, leftrightarrow, "|\cdot|^2"'] & & & G_f(\xi) = \Delta^2\mathrm{sinc}^2(\frac{\Delta}{2}\xi)
\end{tikzcd}
$$

Relation between the ESD and the LSI systems

- The relation between the input and output energy spectral density for LSI systems is given by:

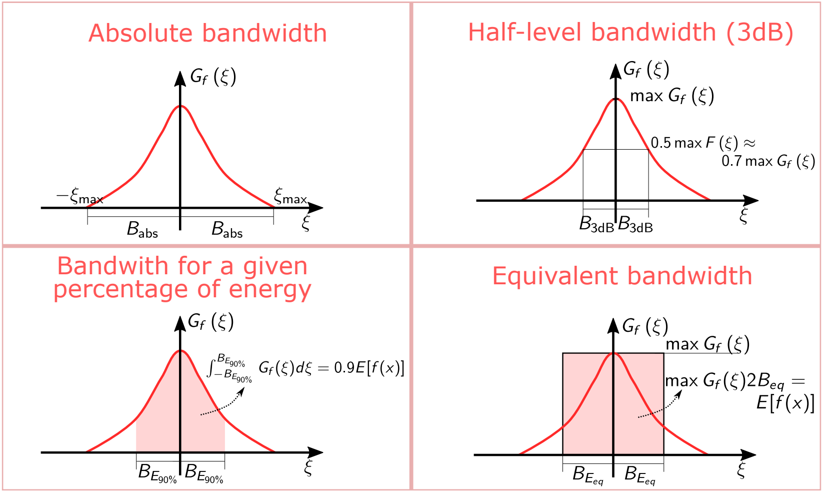
$$\boxed{G_g(\xi) = |H(\xi)|^2\, G_f(\xi)}$$



Outline

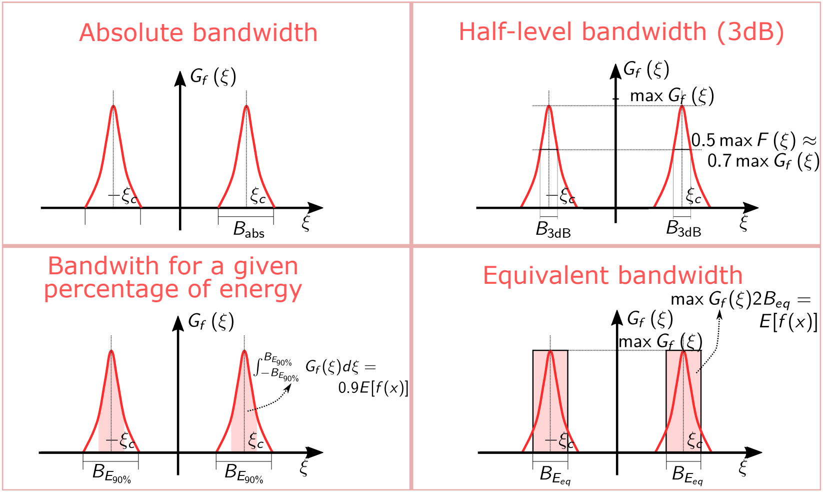
Bandwidth definitions - baseband signals

- In the case of baseband signals (i.e., signals with energy spectral density along $\xi = 0$):



Bandwidth definitions - passband signals

- In the case of passband signals (i.e., signals whose energy spectral density is focused on some spectral point $\xi = \xi_c$):



Outline

Appendixes

Other notations - I

- There are a large number of alternative definitions, which can be preferred depending on the context.
- For example, it is common to replace $i = -j$ in physics:

$$\boxed{\xcancel{F(\xi) = \int_{-\infty}^{\infty} f(x) e^{i\xi x} dx}} \qquad \boxed{\xcancel{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\xi) e^{-i\xi x} d\xi}}$$

- Other authors move or split the factor of 2π:

$$\boxed{\xcancel{F(\xi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-j\xi x} dx}} \qquad \boxed{\xcancel{f(x) = \int_{-\infty}^{\infty} F(\xi) e^{j\xi x} d\xi}}$$

or

$$\boxed{\xcancel{F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-j\xi x} dx}} \qquad \boxed{\xcancel{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{j\xi x} d\xi}}$$

Other notations - II

- Another widespread notation is based on using the conventional frequency $f = \frac{\xi}{2\pi}$ instead of the angular frequency. In these cases, this change of variable cancels out the factor of $\frac{1}{2\pi}$ in both equations:

$$\boxed{\cancel{G(f) = \int_{-\infty}^{\infty} g(x) e^{-j2\pi fx} dx}} \qquad \boxed{\cancel{g(x) = \int_{-\infty}^{\infty} G(f) e^{j2\pi fx} df}}$$

The generalized Fourier Transform - Definition

- Distributions are defined as linear and continuous functionals, which are usually defined by its behavior under the integral sign.
- Under this rigorous definition, it does not make sense to consider the FT.
- Nevertheless, the FT can be generalized of distributions by means of the following observation:
-

$$\int_{-\infty}^{\infty} f(\xi)\mathbf{FT}\left\{g(x)\right\} d\xi = \int_{-\infty}^{\infty} f(\xi)\left(\int_{-\infty}^{\infty} g(x)e^{-j\xi x}dx\right) d\xi =$$
$$\int_{-\infty}^{\infty} g(x)\left(\int_{-\infty}^{\infty} f(\xi)e^{-j\xi x}d\xi\right) dx = \int_{-\infty}^{\infty} \mathbf{FT}\left\{f(\xi)\right\} g(x)dx$$

which is valid for every two (Schwarz) functions.

- By analogy, the generalized FT is defined as:

$$\boxed{\langle f(\xi), \mathbf{FT}\left\{D(x)\right\}\rangle = \langle \mathbf{FT}\left\{f(\xi)\right\}, D(x)\rangle \ \ \forall f(\xi) \in \mathcal{S}\left(\mathbb{R}\right)}$$

- Some authors only use the conventional definition of FT for bounded variation signals in L^1 and they use this definition for the rest of functions (e.g., signals L^2 such as the sinc function).

The generalized Fourier Transform - Examples

- The Fourier transform of $\delta(x)$ can be rigorously defined as:

$$\langle f(\xi), \mathbf{FT}\{\delta(x)\}\rangle = \langle \mathbf{FT}\{f(\xi)\}, \delta(x)\rangle = \mathbf{FT}\{f(\xi)\}\big|_{x=0} = \int_{-\infty}^{\infty} f(\xi) d\xi = \langle f(\xi), 1\rangle$$

and, therefore, $\mathbf{FT}\{\delta(x)\} = 1$.

- Let us consider another example with respect to $\delta'(x)$

$$\left\langle f(\xi), \mathbf{FT}\{\delta'(x)\}\right\rangle = \left\langle \mathbf{FT}\{f(\xi)\}, \delta'(x)\right\rangle = -\left.\frac{d\mathbf{FT}\{f(\xi)\}}{dx}\right|_{x=0}$$

$$= -\int_{-\infty}^{\infty} f(\xi)\, \frac{d}{dx}\left(e^{-j\xi x}\right)\bigg|_{x=0} d\xi = \int_{-\infty}^{\infty} f(\xi)\,(j\xi)\, dk = \langle f(\xi), (j\xi)\rangle$$

and, therefore, $\mathbf{FT}\{\delta'(x)\} = j\xi$.

The generalized Fourier Transform - Examples

- It is possible to prove that the properties of the FT regarding duality, derivatives, shifting, convolutions and parity, which are valid for functions in L^1 can be also applied to distributions.
- For example:

$$\left\langle f(\xi), \mathbf{FT}\left\{D'(x)\right\}\right\rangle = \left\langle \mathbf{FT}\left\{f(\xi)\right\}, D'(x)\right\rangle = -\left\langle \frac{d}{dx}\mathbf{FT}\left\{f(\xi)\right\}, D(x)\right\rangle =$$

$$-\left\langle -j\xi\mathbf{FT}\left\{f(\xi)\right\}, D(x)\right\rangle = j\xi\left\langle \mathbf{FT}\left\{f(\xi)\right\}, D(x)\right\rangle$$

$$= j\xi\left\langle f(\xi), \mathbf{FT}\left\{D(x)\right\}\right\rangle = \left\langle f(\xi), j\xi\mathbf{FT}\left\{D(x)\right\}\right\rangle$$

Time- or space-limited functions

Benedicks' theorem

Benedicks' theorem established that, if $f(x)$ is a well-behave function, then $f(x)$ and $F(\xi)$ cannot have simultaneously finite support.
The result above is a significant generalization of the fact, well known to communication engineers, that a nonzero signal cannot be both time limited and band limited.

Interested readers are referred to the following reference for further details:

- Benedicks, M. (1985), "On Fourier transforms of functions supported on sets of finite Lebesgue measure", *J. Math. Anal. Appl.*, 106 (1): 180–183.