Periodic Signals: the Fourier Series (FS) Representation

Chapter 6

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Outline I

Outline

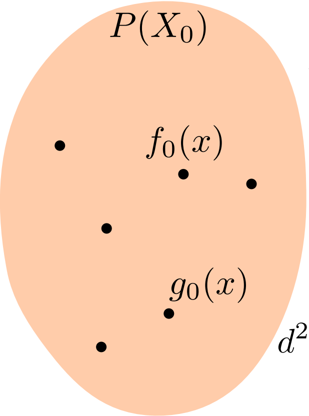
Introduction

- In this chapter, it will be shown how (almost) every periodic function $f_0(x) \in P(X_0)$ can be expressed in terms of a linear combination of the elements of a known basis.
  - The signals of this basis, known as basis functions, are complex exponentials.
  - This linear combination is referred to as Fourier Series (FS).
- The signal space generated by this basis is infinite-dimensional.
  - In practice, the linear combination is truncated to a finite number of terms resulting in some error.
- In addition, it will be shown how signals going through linear and shift-invariant systems can be easily modeled in terms of the FS.

Outline

Overview of the Signal Space

- It is convenient to remember the algebra considered for the signal space $P(X_0)$



$$\langle f_0(x), g_0(x) \rangle = \int_{\langle X_0 \rangle} f_0(x) g_0^*(x) dx$$

$$\| f_0(x) \|^2 = \int_{\langle X_0 \rangle} |f_0(x)|^2 dx$$

$$d^2 \left(f_0(x), g_0(x) \right) = \int_{\langle X_0 \rangle} |f_0(x) - g_0(x)|^2 dx$$

Basis functions

Exercise

Show that the following collection of functions $\{\varphi(x;m)\} = e^{jm\xi_0 x}$:

1. are in $P\left(X_0\right)$
2. have a norm$\left\|\varphi(x;m)\right\|^2 = X_0$
3. are orthogonal, i.e.,

$$\langle\varphi(x;m),\varphi(x;n)\rangle = \begin{cases} X_0 & m = n \\ 0 & m \neq n \end{cases}$$

- Since the basis functions are orthogonal, they are linearly independent and, therefore, they are a basis of some subspace in $P(X_0)$.

Series of complex exponentials

- It is possible to show that the subspace generated by exponentials with imaginary argument

$$\boxed{\{\varphi(x;m)\} = e^{jm\xi_0 x} \text{ with } \xi_0 = \frac{2\pi}{X_0}}$$

covers most of the signals of interest in $P(X_0)$.

- In general, it is possible to represent any CV periodic function as:

$$\boxed{f_0(x) \approx \sum_{m=-\infty}^{\infty} a(m)\varphi(x;m) \text{ with } a(m) = \frac{1}{X_0}\langle f_0(x), \varphi(x;m)\rangle}$$

there are a few exceptions, which will be later discussed.

- Even on these cases, according to the Best Approximation Theorem, this approximation is the best one that can be performed for $f_0(x)$ when using the basis $\{\varphi(x;m)\}$.

Fourier Series

Analysis and Synthesis equations

Taking into account the previous expression, the following analysis and synthesis equations are defined:

$$\boxed{a(m) = \frac{1}{X_0} \int_{\langle X_0 \rangle} f_0(x) e^{-jm\xi_0 x} dx} \text{ (analysis eq.)}$$

$$\boxed{f_0(x) \approx \sum_{m=-\infty}^{\infty} a(m) e^{jm\xi_0 x}} \text{ (synthesis eq.)}$$

which define the Fourier series in the CV case.

Definition in terms of operators

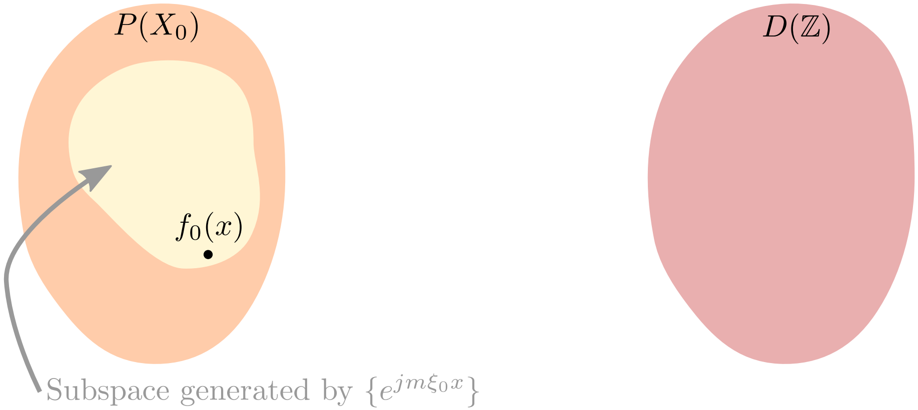
- The FS can be also defined in terms of a (non-invertible) operator between different spaces signals:

$$\begin{aligned}\mathbf{DSF} : P(X_0) &\to D(\mathbb{Z}) \\ f_0(x) &\mapsto a(m) = \frac{1}{X_0}\int_{\langle X_0 \rangle} f_0(x) e^{-jm\xi_0 x} dx\end{aligned}$$

- In a similar way, the operator of the inverse FS is:

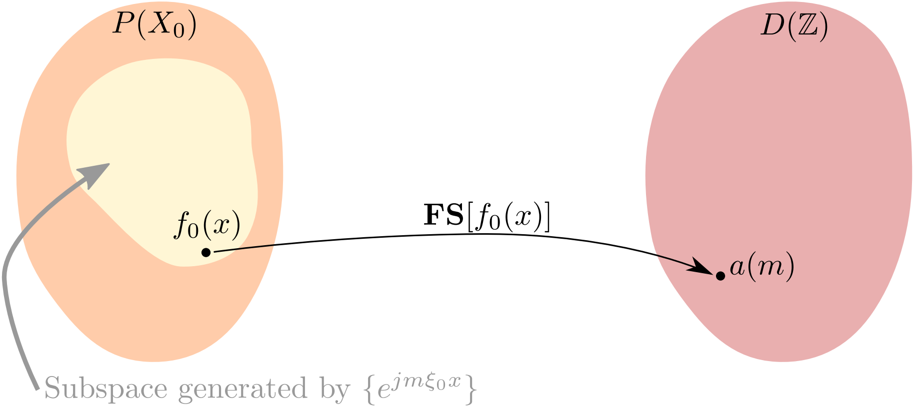
$$\begin{aligned}\mathbf{DSF}^{-1} : D(\mathbb{Z}) &\to P(X_0) \\ a(m) &\mapsto \sum_{m=-\infty}^{\infty} a(m) e^{jm\xi_0 x} \approx f_0(x)\end{aligned}$$

Definition in terms of operators



$$f_0(x) = \mathbf{FS}^{-1}[a(m)] \qquad g_0(x) \neq \mathbf{FS}^{-1}[b(m)]$$

Definition in terms of operators



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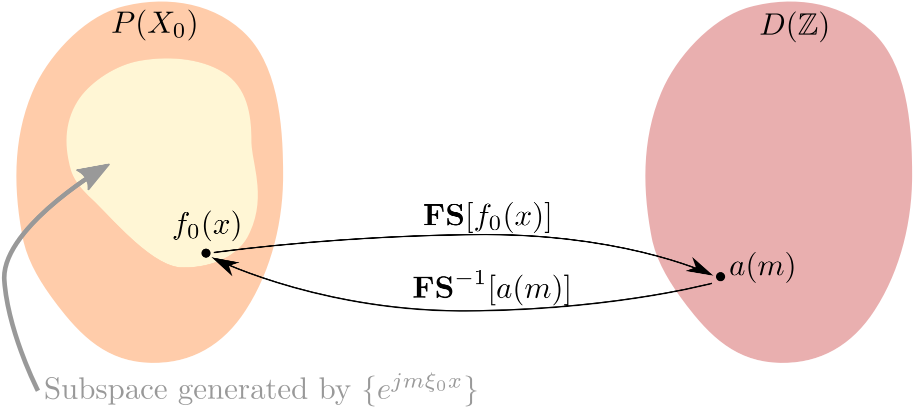
Definition in terms of operators



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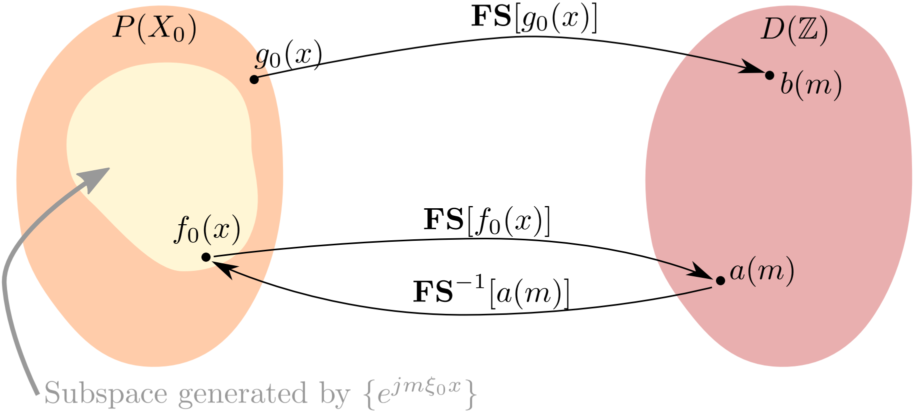

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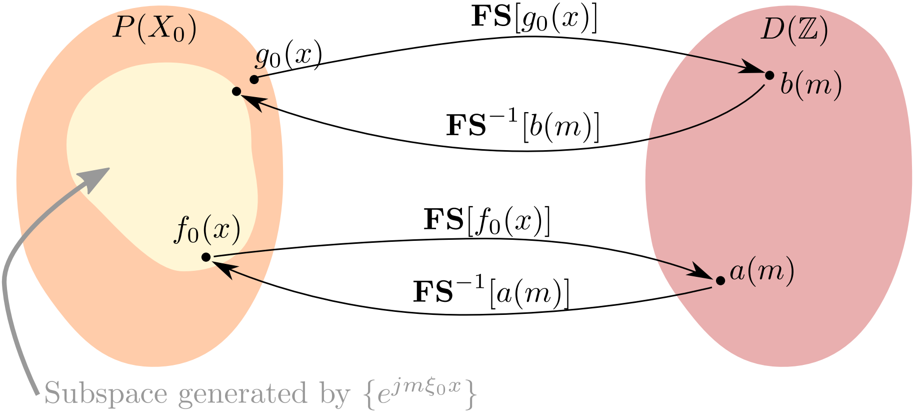

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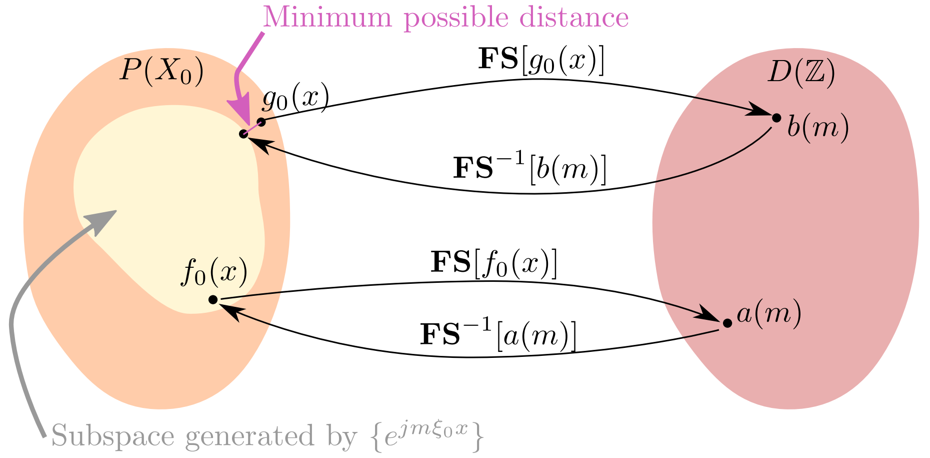

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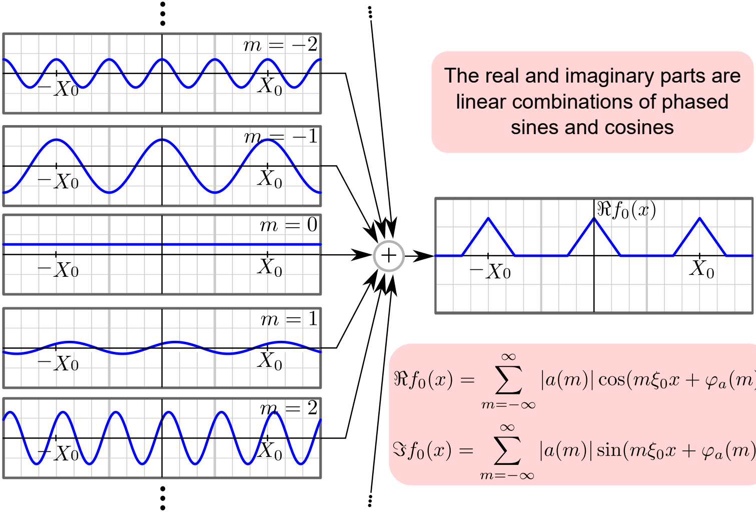
Example: Triangle train



The real and imaginary parts are linear combinations of phased sines and cosines

$$\Re f_0(x) = \sum_{m=-\infty}^{\infty} |a(m)| \cos(m\xi_0 x + \varphi_a(m))$$

$$\Im f_0(x) = \sum_{m=-\infty}^{\infty} |a(m)| \sin(m\xi_0 x + \varphi_a(m))$$

Outline

Truncated FS - I

- In practice, it is not possible to take into account the infinite terms of the synthesis equation:

$$f_{0N}(x) = \sum_{m=-N}^{N} a(m) e^{jm\xi_0 x}$$

- For this reason, it is convenient to analyze the approximation error when only a finite number of terms N is considered

$$d_N^2 = E_0\left[e_{0N}(x)\right] = E_0\left[f_0(x) - f_{0N}(x)\right] = \int_{\langle X_0 \rangle} \left| f_0(x) - f_{0N}(x) \right|^2 dx$$

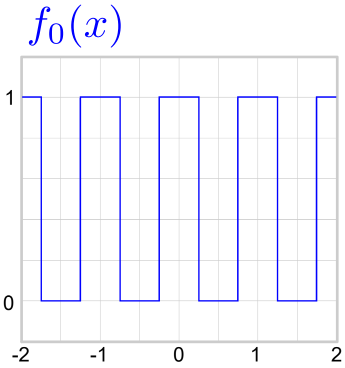
Truncated FS - II

- As a result of the best approximation theorem, this error decreases (or, at least, it does not increase) as N increases.
- Moreover, it is also possible to shown that these coefficients $a(m)$, computed according to the analysis equations, result in an approximation error which tends toward zero.

$$\lim_{N\to\infty} d\left(f_0(x), \sum_{m=-N}^{N} a(m)e^{jm\xi_0 x}\right) = 0$$

as long as these coefficients exists.
- The convergence criteria for $f_{0N}(x)$ will be later detailed.

Graphical analysis of the truncated FS



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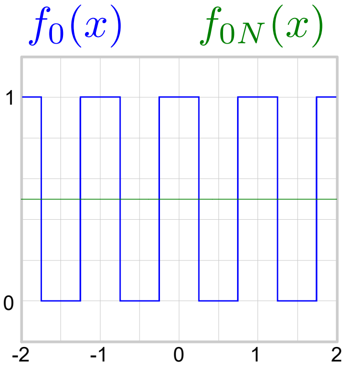
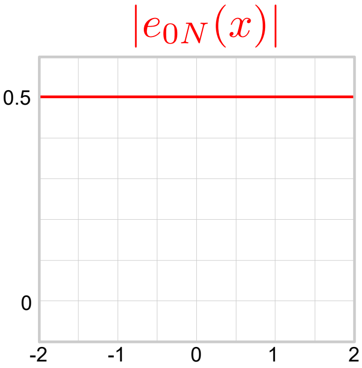


N=0



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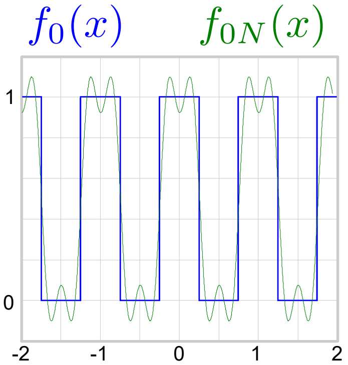
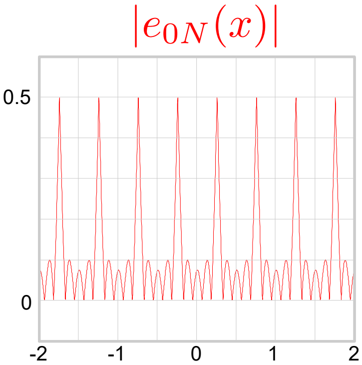


N=3



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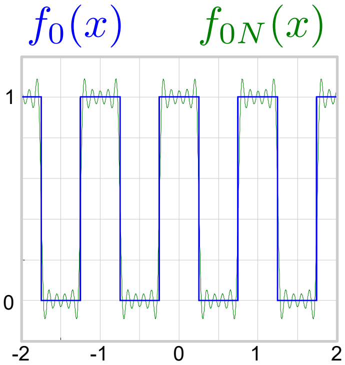
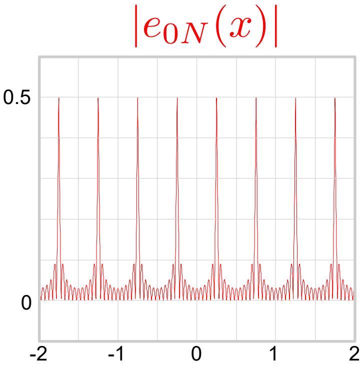


N=10



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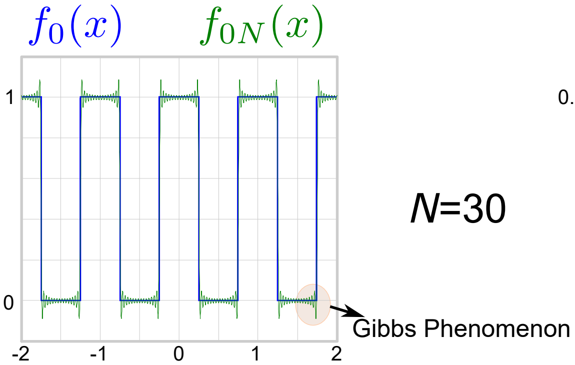
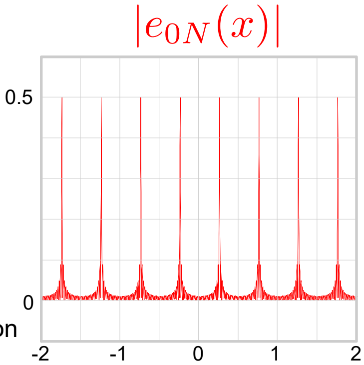


N=30



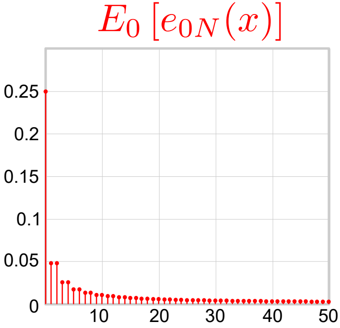
Gibbs Phenomenon

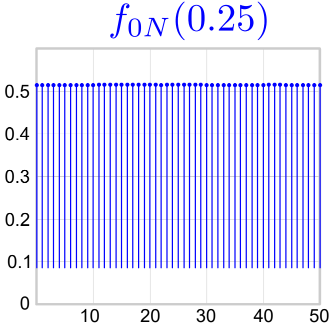
Gibbs Phenomenon

- The truncated FS of discontinuous functions yields functions with large oscillations near the jump discontinuities.
  - If the jump is finite, then the FS converges to the average of the values at either side of the discontinuity.
- The height of error is not reduced by increasing additional terms and, therefore, it does not tend to zero.
  - This oscillation results in an overshoot (i.e., amount exceeding the signal value), which is approximately a 9% for large enough N.
  - Nevertheless, the approximation error d_N does converge toward zero because the area of the error function is reduced.

Convergence of the truncated FS

- In the previous pulse train, note that:
  - despite the approximation error e_{0N} keeps several peaks with a constant value (overshoot), the error energy converges to zero,
  - at the discontinuity (e. g., $x_0 = 0.25$), the FS converges to the average value (0.5).





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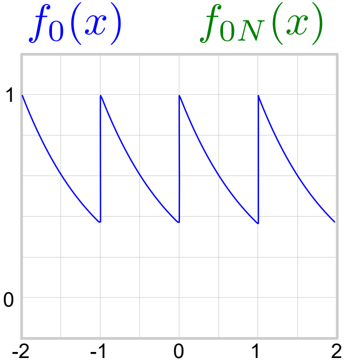

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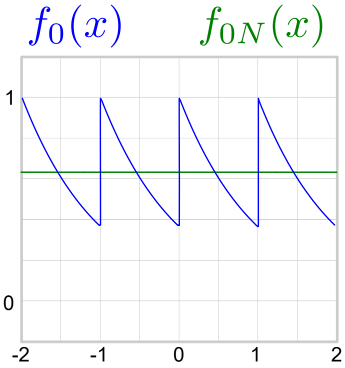
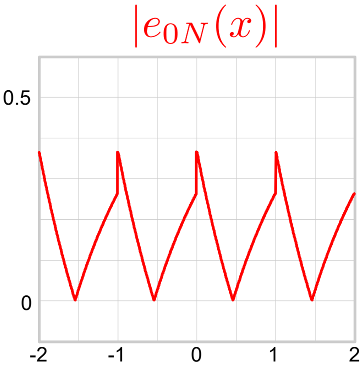


N=0



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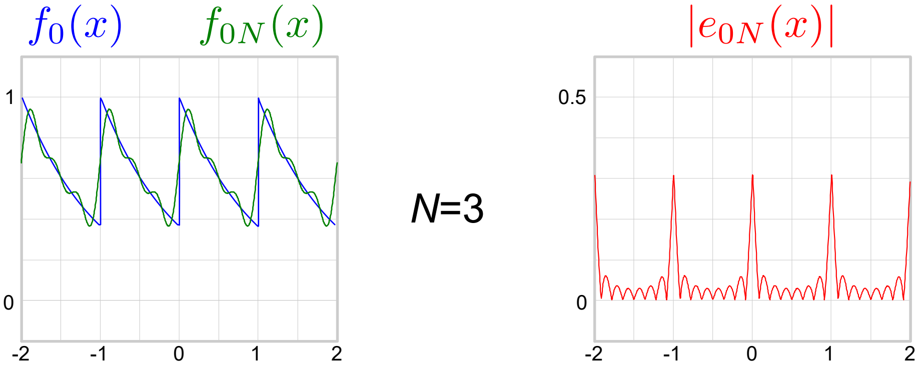
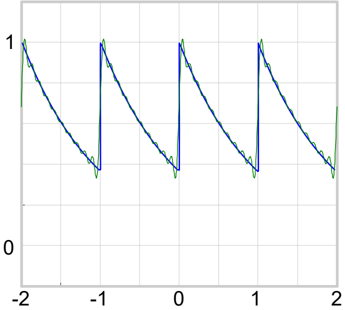


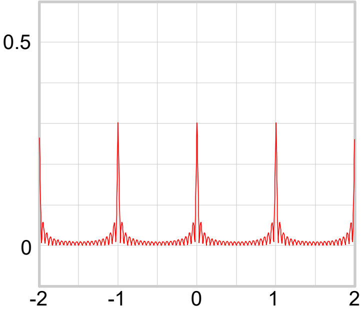
Graphical analysis of the truncated FS

$f_0(x)$ $f_{0N}(x)$



N=10

$|e_{0N}(x)|$



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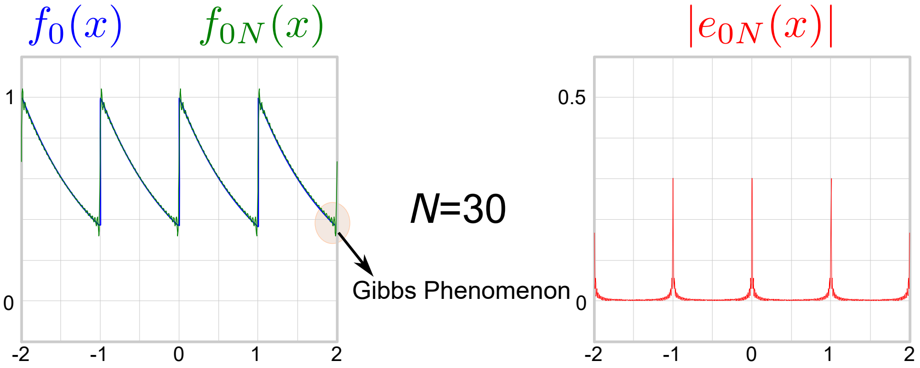
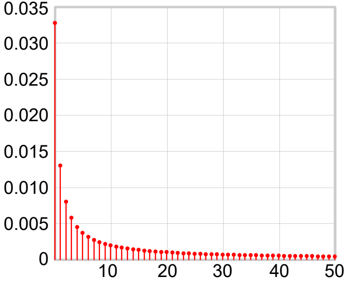


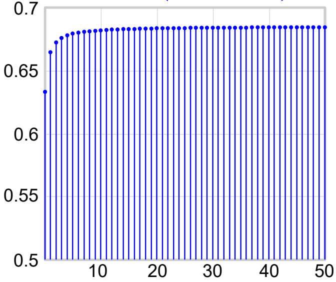
Convergence of the truncated FS

- Again, in the exponential train case, note that:
  - the error e_{0N} keeps several peaks with a constant value (overshoot) but the energy converges to zero,
  - at the discontinuities (e. g., $x_0 = 1$), the FS converges to the average value (0.683).

$E_0\left[e_{0N}(x)\right]$



$f_{0N}(x_0 = 0)$



Criteria for the convergence of FS

- Thus, it seems to be convenient to study the conditions so that the FS exists and converges to the signal under analysis.
- The analysis of the existence and convergence of FS is a really complex matter...
- ... but, in a nutshell, it can be claimed that every signal of interest for engineering can be represented in terms of FS.

> “All reasonable wishes, but starting to ask about convergence of Fourier series, beyond the L^2-convergence, is starting down a road leading to endless complications, details, and, in the end, probably madness.” (Prof. Brad Osgood, Lecture Notes for The Fourier Transform and its Applications, Stanford University)

Concerns with the convergence of FS

- Before studying the converge criteria, the case of a function with non-convergent FS will be considered.
- For this purpose, let us consider the following periodic signal defined over the interval $[0, X_0)$:

$$f_0(x) = \frac{1}{x}, \quad x \in [0, X_0)$$

- The first coefficient of the corresponding FS, $a(0)$, is given by:

$$a(0) = \frac{1}{X_0}\int_0^{X_0} f_0(x)dx = \langle f_0(x)\rangle = \frac{1}{X_0}\int_0^{X_0}\frac{1}{x}dx = \frac{1}{X_0}\ln x\Big|_0^{X_0} \to \infty$$

and, therefore, $f_0(x)$ cannot be represented in terms of a Fourier Series.

Criteria for the convergence of FS

Criteria for convergence

The FS of a signal $f_0(x)$ exists and converges if any of the following conditions is met:

- $f_0(x) \in L_p^2$, i.e., the function is square integrable over a period.
- $f_0(x) \in L_p^1$, i.e., the function is absolutely integrable over a period and, in addition, there are no more than a finite number of a) maxima and minima during any single period b) discontinuities in any finite interval. Moreover, each of these discontinuities is finite.

It is worthwhile to mention that these criteria are sufficient, but not necessary, conditions.

- Each of these criteria yields a different kind of convergence.

Convergence of the FS - Functions L_p^2

- The first convergence criterion corresponds to $f_0(x) \in L_p^2$:

$$\int_{\langle X_0 \rangle} |f_0(x)|^2 dx < \infty$$

- Thus, every signal with finite energy in a period has a FS, which converges in the sense of the 2-norm (norm induced by the inner product):

$$\lim_{N\to\infty} d(f_0(x), f_{0N}(x)) = \lim_{N\to\infty} \|f_0(x) - f_{0N}(x)\| = 0$$

- This fact is particularly relevant since real-world signals have always finite-energy.

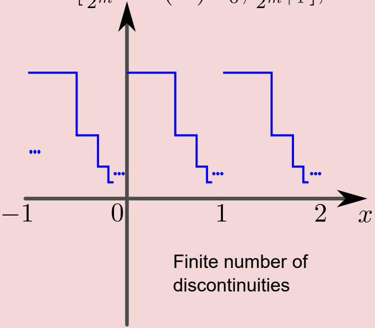
Convergence of the FS - Dirichlet Conditions

- The second convergence criterion is referred to as Dirichlet Conditions:
  1. $f_0(x) \in L_p^1$:
  $$\int_{\langle X_0 \rangle} |f_0(x)| dx < \infty$$
  2. In any finite interval, there are only a finite number of discontinuities. Moreover, each of these discontinuities is finite.
  3. In any finite interval, there are only a finite number of maxima and minima.
- Signals not satisfying the two last conditions are generally pathological and they do not arise in practical contexts.
- If Dirichlet conditions are met, the FS converges to $f_0(x)$ everywhere except at the points of discontinuity.
  - In the discontinuities, the FS converges to the average value at the point of discontinuity.

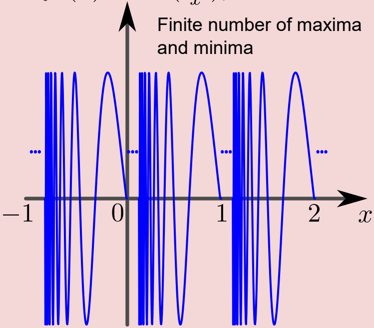
Convergence of the FS - Dirichlet Conditions

$f_0(x) = \frac{1}{2^m}$

$x \in [\frac{X_0}{2^m} - \delta(m)X_0, \frac{X_0}{2^{m+1}}],\ m \in \mathbb{N}$



$f_0(x) = \sin(\frac{2\pi}{x}), 0 < x \leq 1$



Convergence of the FS - Exercise

Exercise

Check if $f_0(x) = \frac{1}{\sqrt{x}}, \quad 0 < x \leq 1$ satisfies any of the convergence criteria.

Convergence of the FS - Exercise

Exercise

Check if $f_0(x) = \frac{1}{\sqrt{x}}, \quad 0 < x \leq 1$ satisfies any of the convergence criteria.

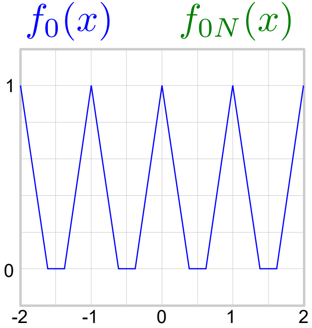
Solution

- It does not meet the first criteria because it is a discontinuous function.
- The second criteria is not met since $\int_0^1 |f_0(x)|^2 dx = \int_0^1 \frac{1}{x} dx = \ln x|_0^1 = \infty$
- The function (a) is absolutely integrable $\int_0^1 |f_0(x)| dx = \int_0^1 \frac{1}{\sqrt{x}} dx = 2\sqrt{x}|_0^1 = 2$, (b) the maxima and minima number is finite, and (c) the number of discontinuities in an interval is finite but the only discontinuity is infinite.

Thus, the existence of the FS is not guaranteed.

Convergence of the FS - Continuous functions

- In the case of continuous periodic signals meeting the "additional" bounded variation conditions, the FS exists always and it converges to the signal.
- Note that the approximation error peaks of the FS do decrease as *N* is increased.
- Moreover, there is not Gibbs phenomenon because the signal is continuous.



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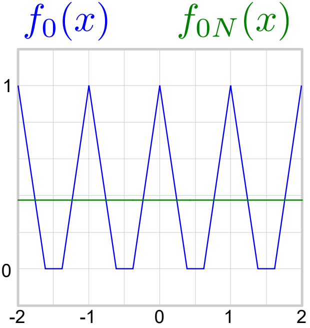
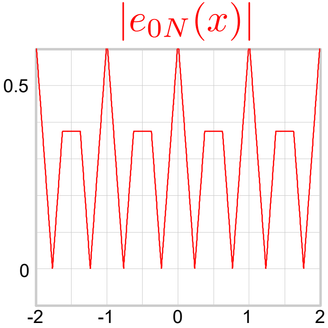


N=0



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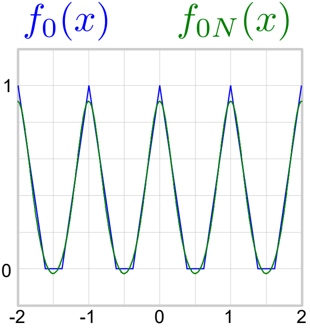
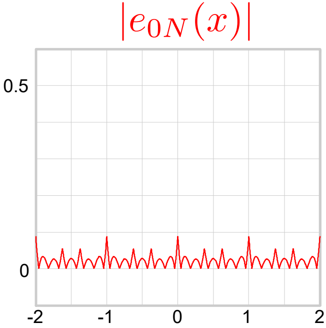


N=3



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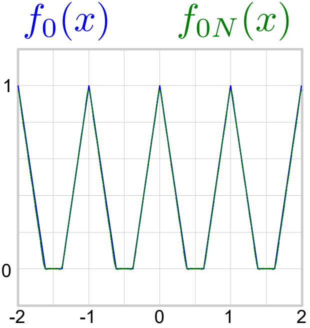
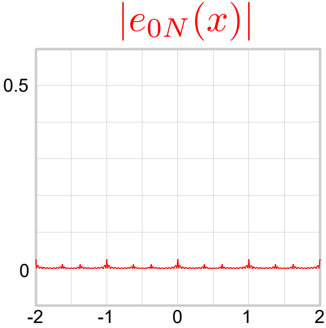


N=10



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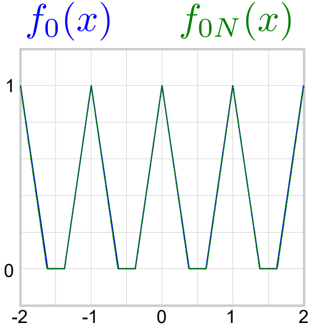
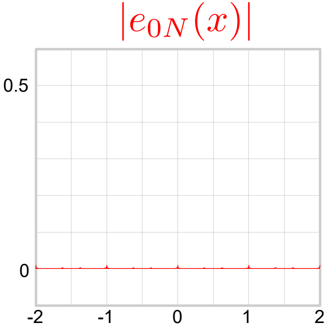


N=30



Outline

General properties of the FS

Linearity	$\alpha f_0(x) + \beta g_0(x)$	$\alpha a(m) + \beta b(m)$
Shifting	$f_0(x - x_o)$	$a(m)e^{-jm\xi_0 x_0}$
Multiplication by the m_0-th harmonic	$f_0(x)e^{jm_0\xi_o}$	$a(m - m_0)$
Conjugation	$f_0^*(x)$	$a^*(-m)$
Reversal	$f_0(-x)$	$a(-m)$
Scaling	$f_0(ax)$, with $a > 0$	$a(m)$ (period $X_0' = \frac{X_0}{a}$)
Periodic convolution	$f_0(x) * g_0(x)$	$X_0 a(m) b(m)$
Multiplication	$f_0(x) g_0(x)$	$a(m) * b(m)$
First derivative	$\frac{d}{dx} f_0(x)$	$jm\xi_0 a(m)$
High order derivatives	$\frac{d^n}{dx^n} f_0(x)$	$(jm\xi_0)^n a(m)$
Integral (only if $\langle f_0(x)\rangle = a(0) = 0$)*	$\int f_0(x)dx$	$\frac{1}{jm\xi_0} a(m)$
Inner product	$\langle f_0(x), g_0(x)\rangle = X_0 \langle a(m), b(m)\rangle$	
Real signals	$f_0(x)$ real	$a(m) = a^*(-m)$

($a(m)$ and $b(m)$ represent the FS coefficients of $f_0(x)$ and $g_0(x)$, respectively)
*Note that $\int f_0(x)dx$ denotes the primitive function of $f_0(x)$. Besides, the constant of integration is supposed to be set to zero

Even/odd signals

In addition to the previous properties, it is interesting to note that:

- The FS of an even signal is an even sequence

$$f_0(x) = f_0(-x) \longleftrightarrow a(m) = a(-m)$$

- The FS of an odd signal is an odd sequence

$$f_0(x) = -f_0(-x) \longleftrightarrow a(m) = -a(-m)$$

- These two properties are very useful to validate results.

Parseval's Theorem

- Taking into account the inner product property, it is trivial to prove the following theorem.

Parseval's (or Plancherel's) Theorem

The energy per period of a periodic signal $f_0(x)$ equals the energy of the FS sequence multiplied by the fundamental period X_0.

$$\boxed{\int_{\langle X_0 \rangle} |f_0(x)|^2 dx = X_0 \sum_{m=-\infty}^{\infty} |a(m)|^2}$$

$$\boxed{E_0\left[f_0(x)\right] = X_0 E\left[a(m)\right]}$$

Example of FS properties

Exercise

(Example 3.9 from the Signals and Systems by A. Oppenheim)
Suppose we are given the following facts about a signal $f_0(x)$:

- is a real-valued signal, whose fundamental period is $X_0 = 4$,
- $a(m) = 0, \quad \forall |m| > 1$,
- The signal with Fourier coefficients $b(m) = e^{-jm\frac{\pi}{2}} a(-m)$ is odd,
- The energy per period is $\int_{<4>} |f_0(x)|^2 dx = 2$.

Example of FS properties

Solution

The second condition entails that only three coefficients $a(-1)$, $a(0)$ and $a(1)$ can be different from zero. In addition, the third condition entails that:

$$b(-1) = -b(1) \Rightarrow e^{j\frac{\pi}{2}} a(1) = -e^{-j\frac{\pi}{2}} a(-1) \Rightarrow \boxed{a(-1) = a(1)}$$

$$b(0) = -b(0) \Rightarrow b(0) = 0 \Rightarrow \boxed{a(0) = 0}$$

Besides, taking into account the Parseval's theorem and the four condition, the following equation is achieved:

$$4\left(|a(-1)|^2 + |a(0)|^2 + |a(1)|^2\right) = 2$$

Since the signal is real-valued, then $|a(-1)| = |a(1)|$, which is in agreement with previous facts, and, therefore, $\boxed{|a(\pm 1)| = \frac{1}{2}}$.

Moreover, a real-valued signal must met $a(m) = a^*(-m)$. If that condition is applied to the coefficient $m = 1$: $a(1) = a^*(-1) = a^*(1)$. Thus, the first coefficient $a(1)$ must be real (and so $a(-1)$). Furthermore, considering the modulus condition, then the following solutions are feasible $a(-1) = a(1) = \pm\frac{1}{2}$.

Thus, the two possible signals meeting all the conditions are:

$$f_0(x) = \pm\cos(\xi_0 x) = \pm\cos\left(\frac{\pi}{2}x\right)$$

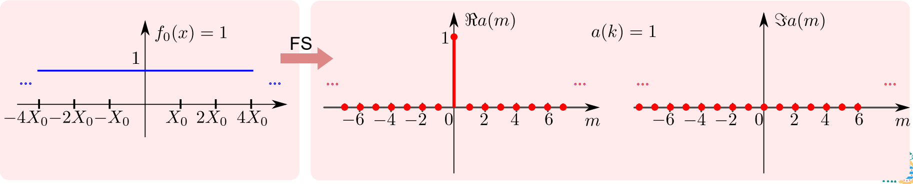
Outline

Sinusoidal signals

- A key piece to calculate complex FS are the harmonic exponentials $f_0(x) = e^{jk\xi_0 x}$.
- In this case, the FS can be find by identifying the coefficients by inspection:

$$\boxed{f_0(x) = e^{jk\xi_0 x} \longleftrightarrow \begin{cases} a(m) = 1 & m = k \\ a(m) = 0 & m \neq k \end{cases}}$$

- A particular case is $k = 0$, which yields a constant function $f_0(x) = 1$.



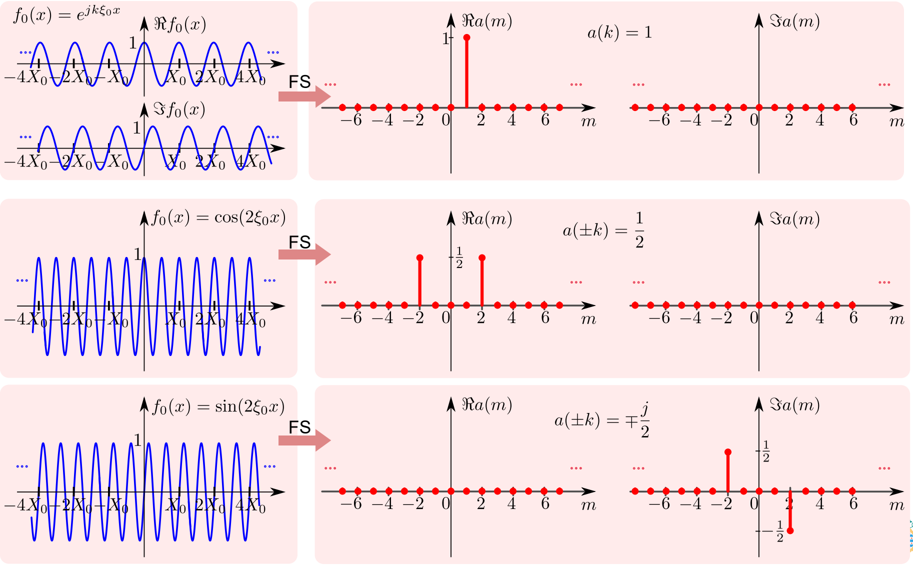
Sinusoidal signals

- Taking into account the previous result, and after decomposing sines and cosines in terms of complex exponentials:

$$\boxed{\cos\left(k\xi_0 x\right) = \frac{e^{jk\xi_0 x} + e^{-jk\xi_0 x}}{2} \longleftrightarrow \begin{cases} a(m) = \frac{1}{2} & m = \pm k \\ a(m) = 0 & m \neq \pm k \end{cases}}$$

$$\boxed{\sin\left(k\xi_0 x\right) = \frac{e^{jk\xi_0 x} - e^{-jk\xi_0 x}}{2j} \longleftrightarrow \begin{cases} a(m) = \frac{1}{2j} & m = k \\ a(m) = \frac{-1}{2j} & m = -k \\ a(m) = 0 & m \neq \pm k \end{cases}}$$

Sinusoidal signals



Sinusoidal signals

Exercise

Calculate the FS representation of the following signal:

$$f_0(x) = 2 + \cos(\xi_0 x) + \cos^2(2\xi_0 x) + \sin(2\xi_0 x)$$

Sinusoidal signals

Exercise

Calculate the FS representation of the following signal:

$$f_0(x) = 2 + \cos(\xi_0 x) + \cos^2(2\xi_0 x) + \sin(2\xi_0 x)$$

Solution

First, the following identity is considered $\cos^2 x = \frac{1+\cos(2x)}{2}$:

$$f_0(x) = 2 + \frac{e^{j\xi_0 x} + e^{-j\xi_0 x}}{2} + \frac{1}{2} + \frac{e^{j4\xi_0 x} + e^{-j4\xi_0 x}}{4} + \frac{e^{j2\xi_0 x} - e^{-j2\xi_0 x}}{2j}$$

Thus, the only nonzero coefficients are:

$$a(0) = \frac{5}{2} \qquad a(\pm 1) = \frac{1}{2} \qquad a(\pm 2) = \pm\frac{1}{2j} \qquad a(\pm 4) = \frac{1}{4}$$

Pulse train

- A very useful signal is the pulse train (also known as periodic square wave)

$$\Pi_{0,\Delta}(x) = \sum_{n=-\infty}^{\infty} \Pi\left(\frac{x - nX_0}{\Delta}\right)$$

whose Fourier Series can be determined from the analysis equation:

$$\boxed{\Pi_{0,\Delta}(x) \longleftrightarrow a(m) = \frac{\Delta}{X_0}\text{sinc}\left(m\pi\frac{\Delta}{X_0}\right)}$$

Proof

$$a(m) = \frac{1}{X_0}\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^{-jm\xi_0 x}dx = \frac{1}{-jm\xi_0 x X_0}\left(e^{-jm\xi_0\frac{\Delta}{2}} - e^{jm\xi_0\frac{\Delta}{2}}\right)$$

$$= \frac{1}{m\pi}\sin\left(m\pi\frac{\Delta}{X_0}\right) = \frac{\Delta}{X_0}\text{sinc}\left(m\pi\frac{\Delta}{X_0}\right)$$

Triangle train

- Using the FS of a pulse train as starting point, it is easy to determine the FS of triangle train

$$\Lambda_{0,\Delta}(x) = \sum_{n=-\infty}^{\infty} \Lambda\left(\frac{x - nX_0}{\Delta}\right)$$

whose FS representation is given by:

$$\boxed{\Lambda_{0,\Delta}(x) \longleftrightarrow a(m) = \frac{\Delta}{2X_0}\mathrm{sinc}^2\left(m\pi\frac{\Delta}{2X_0}\right)}$$

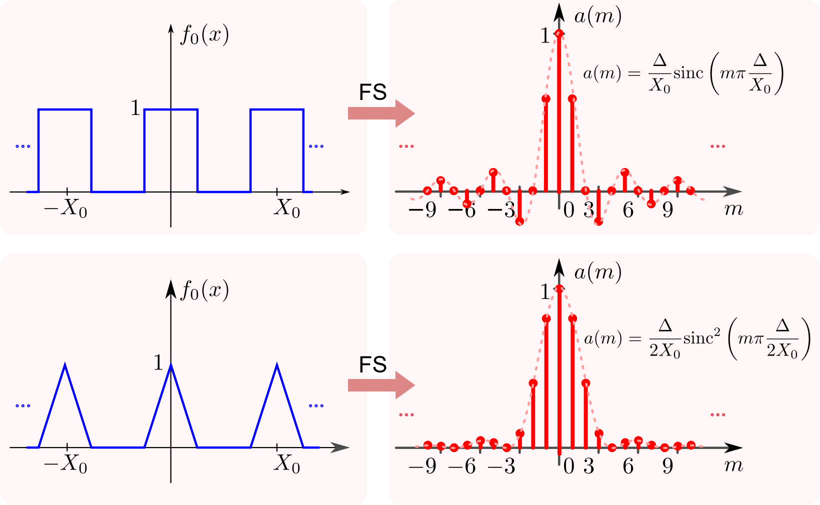
Proof

Taking into account the relationship between pulses and triangles, and the periodic convolution property:

$$\Lambda\left(\frac{x}{\Delta}\right) = \frac{2}{\Delta}\Pi\left(\frac{x}{\Delta/2}\right) * \Pi\left(\frac{x}{\Delta/2}\right) \Rightarrow \Lambda_{0,\Delta}(x) = \frac{2}{\Delta}\Pi_{0,\frac{\Delta}{2}}(x) * \Pi_{0,\frac{\Delta}{2}}(x)$$

$$\Lambda_{0,\Delta}(x) \longleftrightarrow X_0\frac{2}{\Delta}\left[\frac{\Delta}{2X_0}\mathrm{sinc}\left(m\pi\frac{\Delta}{2X_0}\right)\right]^2 = \frac{\Delta}{2X_0}\mathrm{sinc}^2\left(m\pi\frac{\Delta}{2X_0}\right)$$

Triangle train



Exponential train

Exercise

Taking into account the definition of FS, determine the coefficients of the FS of an exponential train defined as:

$$f_0(x) = e^{-\frac{ax}{X_0}}, \quad x \in [0, X_0)$$

Solution

$$a(m) = \frac{1}{X_0} \int_{\langle X_0 \rangle} e^{-\frac{ax}{X_0}} e^{-jm\xi_0 x} dx = \frac{1}{X_0} \int_0^{X_0} e^{-\left(\frac{a}{X_0} + jm\xi_0\right)x} dx =$$

$$= \frac{1}{X_0} \frac{-1}{\frac{a}{X_0} + jm\xi_0} \left[e^{-\left(\frac{a}{X_0} + jm\xi_0\right)x} \right]_{x=0}^{X_0} = \frac{1}{a + jm2\pi} \left[1 - e^{-(a + jm\xi_0 X_0)} \right]$$

$$= \frac{1 - e^{-a}}{a + jm2\pi}$$

Periodic distributions

- A distribution $D_0(x)$ is said to be periodic with period X_0 if

$$\exists X_0 \text{ t.q. } \int_{-\infty}^{\infty} f(x) D_0(x) dx = \int_{-\infty}^{\infty} f(x) D_0(x+X_0) dx, \quad \forall f(x)$$

- The concept of FS for distributions requires a complex framework.
- Nevertheless, it can be shown that in the case of impulse-like distributions (Dirac delta, its derivatives) but also, in general, any bounded support distribution, it is possible to use regular operations using the definition of FS and the properties of the corresponding distribution.
- The following trains of periodic distributions will be considered:

$$\delta_0(x) = \sum_{n=-\infty}^{\infty} \delta(x - nX_0) \qquad \delta_0'(x) = \sum_{n=-\infty}^{\infty} \delta'(x - nX_0)$$

Periodic distributions

$$\boxed{\delta_0(x) \longleftrightarrow a(m) = \frac{1}{X_0}}$$

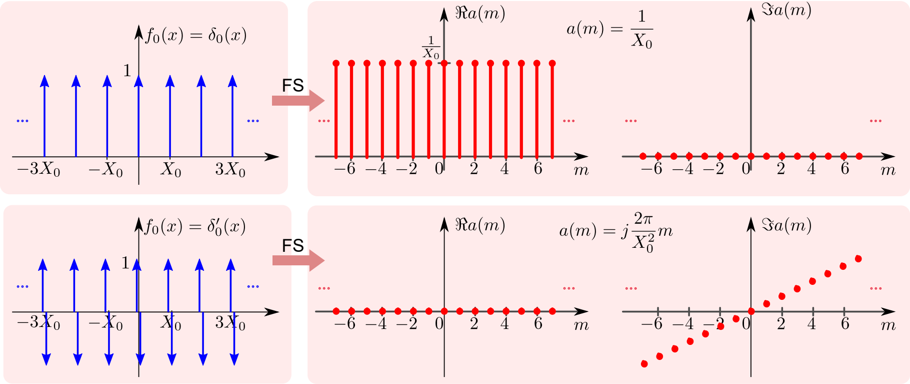
$$\boxed{\delta_0'(x) \longleftrightarrow a(m) = j\frac{2\pi}{X_0^2}m}$$

Proof

$$\delta_0(x) \longleftrightarrow a(m) = \frac{1}{X_0}\int_{-\frac{X_0}{2}}^{\frac{X_0}{2}} \delta(x) e^{-jm\xi_0 x} dx = \frac{1}{X_0}$$

$$\delta_0'(x) \longleftrightarrow a(m) = \frac{1}{X_0}\int_{-\frac{X_0}{2}}^{\frac{X_0}{2}} \delta'(x) e^{-jm\xi_0 x} dx = -\frac{-jm\xi_0}{X_0} = j\frac{2\pi}{X_0^2}m$$

Periodic distributions



Periodic distributions

Exercise

Compute the FS for a train of second order derivatives of the Dirac delta $\delta_0''(x)$ resorting to (a) the definition of $\delta''(x)$ and (b) the high-order derivative properties of the FS.

Exercise

Sketch the following signal, determine its fundamental period in terms of X_0' and calculate the FS:

$$f_0(x) = \sum_{m=-\infty}^{\infty} (-1)^m \delta(x - mX_0')$$

Exercise

Calculate the derivative in the sense of distributions of the following pulse train:

$$f_0(x) = \sum_{m=-\infty}^{\infty} \Pi\left(\frac{x - mX_0 - \frac{X_0}{4}}{\frac{X_0}{2}}\right)$$

and calculate the FS for the previous switching Dirac Delta train by exploiting this last result.

Periodic distributions

Solution

Considering the FS definition:

$$a(m) = \frac{1}{X_0}\int_{\langle X_0\rangle} \delta_0''(x)e^{-jm\xi_0 x}dx = \frac{1}{X_0}\int_{-\frac{X_0}{2}}^{\frac{X_0}{2}} \delta''(x)e^{-jm\xi_0 x}dx = \frac{(-jm\xi_0)^2}{X_0}$$

$$a(m) = -\frac{4\pi^2}{X_0^3}m^2$$

Periodic distributions

Solution

The period of the signal is $X_0 = 2X_0'$. The signal can be easily decomposed into two impulse trains with period X_0:

$$f_0(x) = \sum_{m=-\infty}^{\infty} \delta\left(x - 2mX_0'\right) - \sum_{m=-\infty}^{\infty} \delta\left(x - (2m+1)\,X_0'\right) =$$

$$= \sum_{m=-\infty}^{\infty} \delta\left(x - mX_0\right) - \sum_{m=-\infty}^{\infty} \delta\left(x - mX_0 - \frac{X_0}{2}\right)$$

Taking into account the properties for sums and shifts::

$$a(m) = \frac{1}{X_0} - \frac{1}{X_0} e^{-jm\xi_0 \frac{X_0}{2}} = \frac{1}{X_0} - \frac{1}{X_0} e^{-jm\pi} = \frac{1}{X_0}\left(1 - (-1)^m\right) = \begin{cases} 0 & m \text{ even} \\ \frac{2}{X_0} & m \text{ odd} \end{cases}$$

Poisson summation formula

Poisson summation formula

The FS of an impulse train enables us to obtain the Poisson summation formula:

$$\boxed{\sum_{m=-\infty}^{\infty} \delta(x - mX_0) = \frac{1}{X_0} \sum_{m=-\infty}^{\infty} e^{jm\xi_0 x}}$$

- This result is very useful because it enables us to establish a connection between a Dirac comb (i.e., impulse train) and a sum of harmonically related complex exponentials.

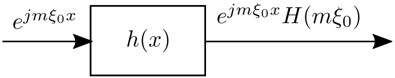
Outline

Response of a LSI system to a complex exponential

- As it has been described, a periodic signal can be generally represented by a linear combination of complex exponentials $\{\varphi(x;m)\} = e^{jm\xi_0 x}$
- Thus, it is convenient to study the response of a LSI system to a complex exponential:

$$\mathbf{F}\left[\varphi(x;m)\right] = \varphi(x;m) * h(x) = \int_{-\infty}^{\infty} h(x')e^{jm\xi_0(x-x')}dx' =$$
$$e^{jm\xi_0 x}\int_{-\infty}^{\infty} h(x')e^{-jm\xi_0 x'}dx'$$

- Thus, the complex exponentials are eigenfunctions of the LSI systems.



$$\boxed{H(\xi) = \int_{-\infty}^{\infty} h(x)e^{-j\xi x}dx}$$

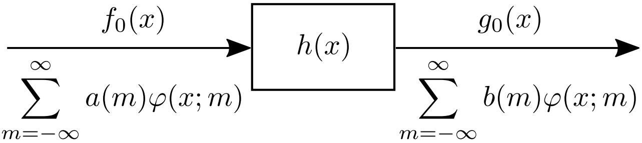
Response of a LSI system to a periodic signal

- Consequently, the response of a LSI system to a periodic signal $f_0(x)$, if there FS exists, is:

$$g_0(x) = \mathbf{F}\left[f_0(x)\right] = \sum_{m=-\infty}^{\infty} a(m)\mathbf{F}\left[\varphi(x;m)\right] = \sum_{m=-\infty}^{\infty} b(m)\varphi(x;m)$$

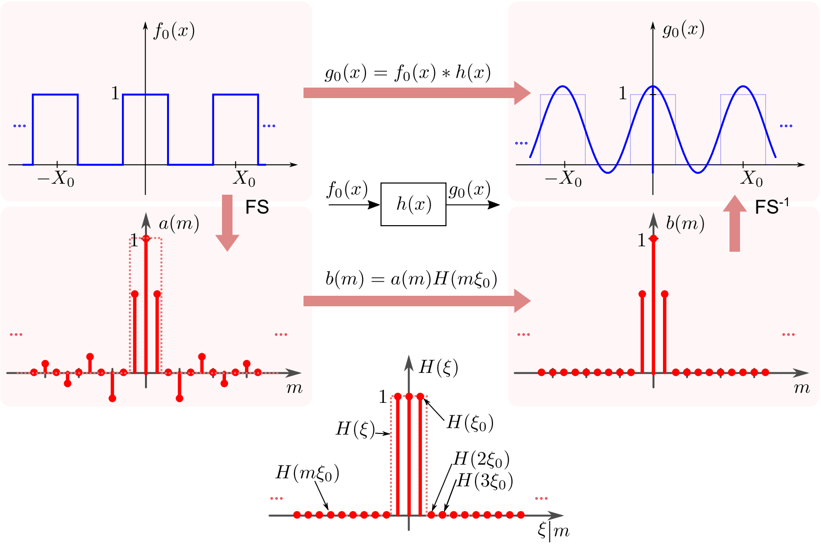
- Thus, the output signal is also a periodic signal with identical period, whose FS coefficients are given by

$$\boxed{b(m) = a(m)H(m\xi_0)}$$



- The term $H(\xi) = \int_{-\infty}^{\infty} h(x)e^{-j\xi x}dx$ is known as Fourier Transform of $h(x)$ and it will be studied in the next chapter.

Response of a LSI system to a periodic signal



Outline

Appendixes

Other FS notations

- In some works, a slightly different definition of the Fourier series is considered:

$$f_0(x) \approx \frac{\alpha(0)}{2} + \sum_{m=1}^{\infty} \left[\alpha(m)\cos\left(m\xi_0 x\right) + \beta(m)\sin\left(m\xi_0 x\right)\right]$$

where the coefficients are given by:

$$\alpha(m) = \frac{2}{X_0}\int_{\langle X_0\rangle} f_0(x)\cos\left(m\xi_0 x\right)dx \qquad \beta(m) = \frac{2}{X_0}\int_{\langle X_0\rangle} f_0(x)\sin\left(m\xi_0 x\right)dx$$

- This definition is preferred in some contexts because if $f_0(x)$ is real-valued, then it only requires the use real-valued functions and real coefficients.
- The relations between both notations are:

$$\boxed{\alpha(m) = a(m) + a(-m) \qquad \beta(m) = \frac{a(m) - a(-m)}{j}}$$

$$\boxed{a(-|m|) = \frac{\alpha(m) - j\beta(m)}{2} \qquad a(|m|) = \frac{\alpha(m) + j\beta(m)}{2}}$$

Convergence rate

- The "smoother" the function, the less variations it suffers from.
- If the function changes slowly, then it makes sense that the high frequency terms will have a low weight in the FS.
- In practice, the "smoothness" of functions is measured in terms of the differentiability.
- It is possible to show that:
  - If $f(x) \notin C^0$, then the Fourier coefficients $a(m)$ should have some terms like $\frac{1}{m}$.
    - For example, a pulse train.
  - If $\exists f'(x)$ (almost everywhere) but $f'(x) \notin C^0$, then the Fourier coefficients $a(m)$ should have some terms like $\frac{1}{m^2}$.
    - For example, a triangle train.
  - If $\exists f''(x)$ (almost everywhere) but $f''(x) \notin C^0$, then the Fourier coefficients $a(m)$ should have some terms like $\frac{1}{m^3}$.