The relation between the Dirac Delta and the linear systems

Chapter 5 - Part II

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Outline I

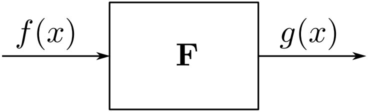
1. Introduction
2. Representing signals in terms of the Dirac Delta
3. Dirac Delta and linear systems
   - Linear systems
   - Linear and shift-invariant systems - Impulse response
   - LSI systems properties in terms of the impulse response impulse

Outline

1. Introduction
2. Representing signals in terms of the Dirac Delta
3. Dirac Delta and linear systems

Introduction

- Dirac Delta plays a key role in characterizing linear systems.
- It is convenient to remember that the relation between input and output in a linear system is given by:



$$g(x) = \mathbf{F}(f(x))$$

- where $\mathbf{F}(\cdot)$ is a linear operator and, therefore,

$$\mathbf{F}\left(\alpha f_1(x) + \beta f_2(x)\right) = \alpha \mathbf{F}\left(f_1(x)\right) + \beta \mathbf{F}\left(f_1(x)\right)$$

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Representation of signals in $S(\mathbb{R})$ in terms of Dirac Deltas

- On one hand, according to the Dirac Delta properties:

$$f(x) = f(x) * \delta(x) = \int_{-\infty}^{\infty} f(x - x')\delta(x')dx' = \int_{-\infty}^{\infty} f(x')\delta(x - x')dx'$$

- On the other hand, it has been shown that considering an uncountable base, a signal can be expanded by means of a generalized linear combination:

$$f(x) = \int_{-\infty}^{\infty} \alpha(x')e(x;x')dx'$$

comparing both equations an important result can be inferred.

Dirac Deltas as a basis

The collection of distributions $\{\delta(x - x')\}$ enables us to represent any function and, therefore, it plays a role equivalent to the role of a *basis* of $S(\mathbb{R})$

$$\boxed{e(x;x') = \delta(x - x')}$$

where the weights of the linear combination representing a function $f(x)$ are given by:

$$\boxed{\alpha(x') = f(x')}$$

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Linear systems

- Taking into account the previous representation, the output of a linear system is given by:

$$g(x) = \mathbf{F}(f(x)) = \mathbf{F}\left(\int_{-\infty}^{\infty} f(x')\delta(x-x')dx'\right) =$$
$$\int_{-\infty}^{\infty} f(x')\mathbf{F}\left(\delta(x-x')\right)dx' = \int_{-\infty}^{\infty} f(x')h(x;x')dx'$$

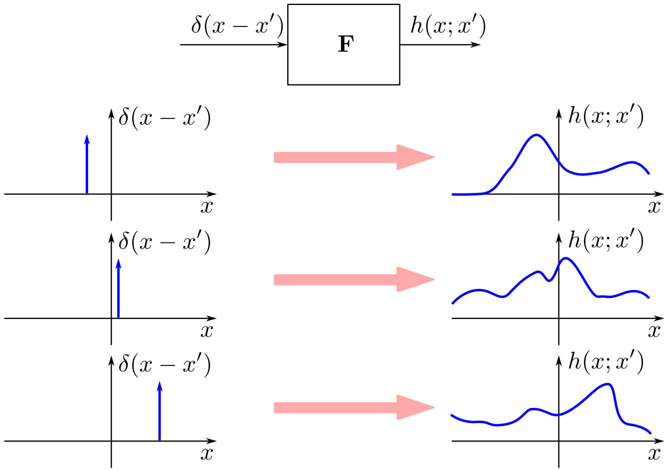
Collection of impulse responses

A linear operator **F** is fully characterized by the following integral operator:

$$\mathbf{F}(\cdot) = \int_{-\infty}^{\infty} (\cdot)\, h(x;x')dx'$$

where $\boxed{h(x;x') = \mathbf{F}\left(\delta(x-x')\right)}$ is the collection of impulse responses.

Linear systems



- Note that the impulse response generally depends on the impulse position.

Linear systems

Exercise

Compute the collection of impulse responses $h(x;x')$ of the following systems:

1. $g(x) = \begin{cases} f(x) & x \geq 0 \\ 0 & x < 0 \end{cases}$
2. $g(x) = \int_{-\infty}^{\infty} f(y) e^{-(x-2y)^2} dy$
3. $g(x) = \int_{-\infty}^{\infty} f(x-y) e^{-y^2} dy$

Solution

1. $h(x;x') = \begin{cases} \delta(x-x') & x' \geq 0 \\ 0 & x' < 0 \end{cases} = u(x)\delta(x-x') = u(x')\delta(x-x')$
2. $h(x;x') = \int_{-\infty}^{\infty} \delta(y-x') e^{-(x-2y)^2} dy = e^{-\left(x-2x'\right)^2}$
3. $h(x;x') = \int_{-\infty}^{\infty} \delta(x-x'-y) e^{-y^2} dy = e^{-\left(x-x'\right)^2}$

Linear and shift-invariant systems

- In a linear and shift-invariant (LSI) system, the impulse response will be always the same one regardless of the impulse position, except for the corresponding shift:

$$\delta(x) \rightarrow h(x) = \mathbf{F}\left(\delta(x)\right)$$
$$\delta(x-x') \rightarrow h(x;x') = \mathbf{F}\left(\delta(x-x')\right) = h(x-x'))$$

Impulse response

Consequently, the output of a LSI is given by:

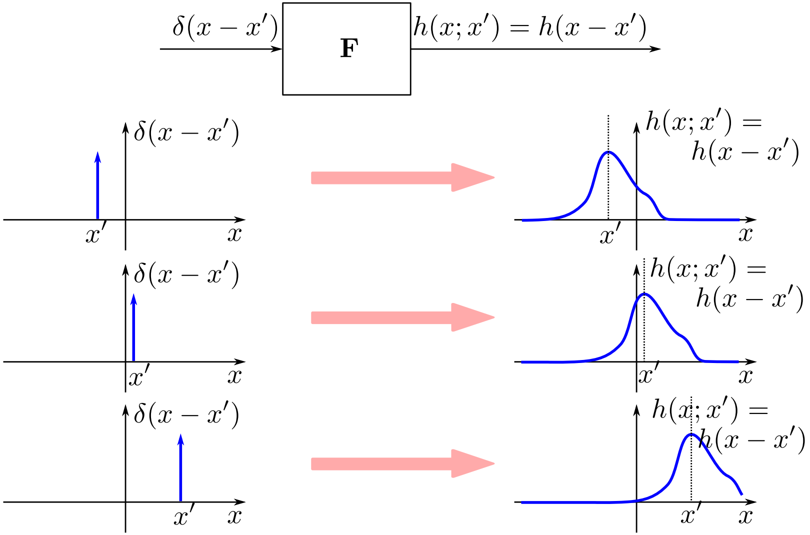
$$g(x) \;=\; \mathbf{F}\left(f(x)\right) \;=\; \int_{-\infty}^{\infty} f(x')h(x;x')dx' \;=\; \int_{-\infty}^{\infty} f(x')h(x \;-\; x')dx'$$

which can be rewritten as

$$\boxed{g(x) = f(x) * h(x)}$$

where $h(x)$ is the impulse response of the system.

Linear and shift-invariant systems



Properties of the LSI systems

Properties of the LSI systems

- As it has been previously shown, the output of a LSI system is given by the convolution of the input and impulse response:

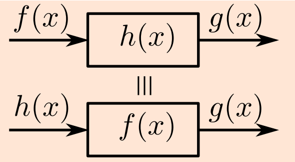
$$g(x) = f(x) * h(x)$$

  and, therefore, LSI systems inherit the properties from the convolution operation.
- In addition, the properties of the LSI systems (memory, causality, stability, etc.) can be determined from the impulse response.

Interconnection of LSI systems

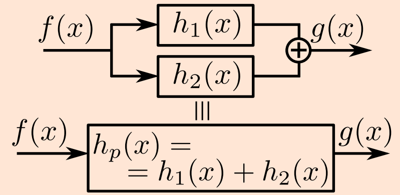
Commutativity

$$f(x) * h(x) = h(x) * f(x)$$



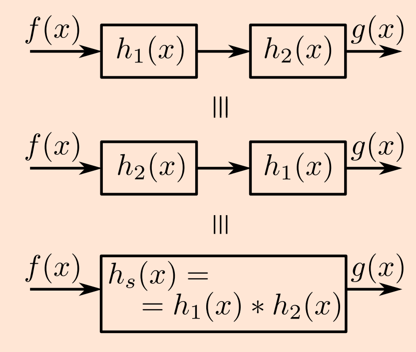
Distributivity

$$f(x) * h_1(x) + f(x) * h_2(x) = \\ = f(x) * (h_1(x) + h_2(x))$$



Associativity

$$(f(x) * h_1(x)) * h_2(x) = \\ (f(x) * h_2(x)) * h_1(x) = \\ f(x) * (h_1(x) * h_2(x))$$



Properties of the LSI systems - Memory

- Since a LSI system is fully characterized from its impulse response $h(x)$, one might wonder what kind of impulse response characterizes a memoryless system.

$$g(x_0) = \int_{-\infty}^{\infty} f(x')h(x_0 - x')dx'$$

- It is easy to infer from the previous equation that the output at a position x_0 requires generally all the values of the input $f(x)$ from $-\infty$ to $+\infty$.
- If only the value $f(x_0)$ were used for any function $f(x)$ and any position x_0, then the function $h(x_0 - x')$ should have a support bounded to a single point $x' = x_0$.
- The only "function" satisfying that property, and not yielding a zero output, is the Dirac Delta, which might be multiplied by a scalar:

$$h(x_0 - x') = K\delta(x_0 - x')$$

Properties of the LSI systems - Memory

Memory

The impulse response of a memoryless LSI system has the form:

$$\boxed{h(x) = K\delta(x), \quad K \in \mathbb{C}}$$

Properties of the LSI systems - Causality

Causality

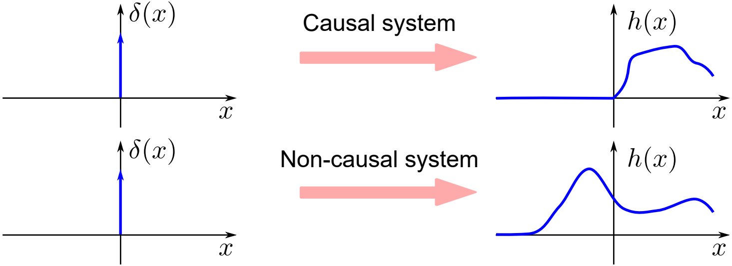
A SLI causal, if the output at every position depends only on the input at the current or previous positions:

$$g(x_0) = \int_{-\infty}^{\infty} f(x')h(x_0 - x')dx' = \int_{-\infty}^{x_0} f(x')h(x_0 - x')dx'$$

the previous conditions is satisfy $\forall f(x)$ and $\forall x_0$ if and only if $h(x_0 - x') = 0,\ \forall x' > x_0$. Thus, the impulse response has the form:

$$\boxed{h(x) = 0,\ \forall x < 0}$$

Properties of the LSI systems - Causality



Example

The system $h(x) = \delta(x - x_0)$ is causal if and only if $x_0 \geq 0$

Properties of the LSI systems - Stability

- The stability property will be proved in a problem solving session.

Stability

A LSI system is stable if and only if:

$$\boxed{\text{SLI stable} \Leftrightarrow \int_{-\infty}^{\infty} |h(x)|dx < \infty}$$

that is, the 1-norm of the impulse response is finite, $h(x) \in L^1$ (absolutely integrable signal).

- Note: the sinc function, though not intuitive, is not absolutely integrable (there is an erratum in the book by E. Gago about this).

Properties of the LSI systems - Stability

Exercise

Let us consider the system described by the following equation (i.e., an integrator):

$$g(x) = \int_{-\infty}^{x} f(x')dx'$$

1. Determine whether or not it is a LSI system.
2. If it is a LSI system, find the impulse response.
3. Determine whether or not it is stable.

Invertibility

Invertibility

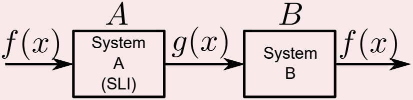
- It is possible to shown (see next exercise) that if a LSI is invertible, then the inverse system is also a LSI
- Such an inverse system $h^{-1}(x)$ satisfies the following property:

$$\boxed{h(x) * h^{-1}(x) = \delta(x)}$$

Invertibility

Exercise

(Based on exercise 2.50 from book by Oppenheim)
Consider the cascade of two systems, A y B. The first system, A, is known to be LSI. The second system, B, is known to be the inverse of system A. Let $g_1(x)$ and $g_2(x)$ denote the output of A to the inputs $f_1(x)$ and $f_2(x)$, respectively:



1. What is the response of system B to $\alpha g_1(x) + \beta g_2(x)$?
2. What is the response of system B to $g(x - x_0)$?