The Dirac Delta

Chapter 5

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Outline I

Outline

Introduction

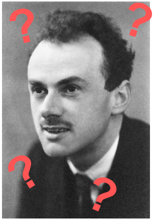
- Paul Dirac, Nobel Prize in Physics (quantum mechanics), searched for a function $g(x)$ meeting the following integral equation for every function $f(x)$

$$\int_{-\infty}^{\infty} f(x)g(x)dx = f(0)$$

  discovering that such a function is not possible.

- Thus, he decided to build a "mathematical object", which was an "improper function" according to his terminology.

- Laurent Schwartz, French mathematician awarded the Fields medal, gave well-defined meaning to those objects by means of the Theory of Distributions.



$$\int_{-\infty}^{\infty} f(x)g(x)dx = f(0)$$

$$\boxed{\delta(x)}$$

Outline

Definition

Dirac Delta

Dirac Delta is not a real function but a "mathematical objects" whose behavior is only defining when operating under the integral sign:

$$\boxed{\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)}$$

- These "mathematical object" is rigorously defined by means of the Theory of Distributions.

- In this course, we will focus on the practical aspects from an engineering point of view and, therefore, the most abstract details will be dodged:
  - This description corresponds to the conventional one in Physics and engineering.

Transformations of the independent variable

- It is possible to show that this function-like notation $\delta(x)$, though it is not preferred in mathematics, it enables to agilely evaluate a large number of problems in engineering.
  - This notation is compatible with conventional scaling, reversal and shifts of the independent variable.

Transformations of the independent variable

By means of changes of variable, it is simple to show that:

$$\int_{-\infty}^{\infty} f(x)\delta(ax - x_0)dx = \frac{1}{|a|} f\left(\frac{x_0}{a}\right) = \int_{-\infty}^{\infty} f(x)\left[\frac{1}{|a|}\delta\left(x - \frac{x_0}{a}\right)\right] dx$$

Hence, both generalized functions (also known as distributions) have the same behavior under the integral sign:

$$\boxed{\delta(ax - x_0) = \frac{1}{|a|}\delta\left(x - \frac{x_0}{a}\right)}$$

(this equality in only meaningful in the sense of distributions)

Outline

Dirac delta as a limit of a sequence of functions

- Although there is not any function such that

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0), \quad \forall f(x)$$

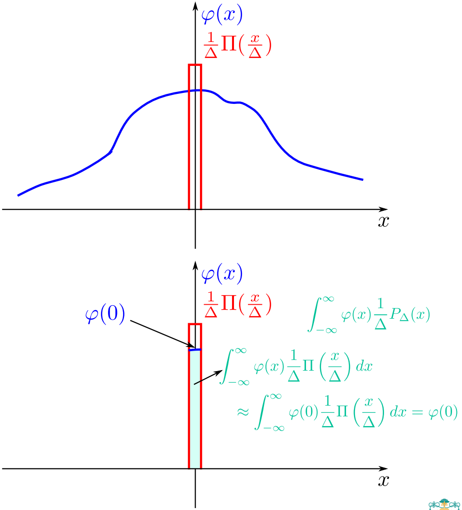
it is easy to show that for every $f(x)$:

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\left[n\Pi(nx)\right]dx = f(0).$$

- It is said that the following sequence of functions

$$\boxed{p_n(x) = n\Pi(nx)}$$

converges to $\delta(x)$ in the sense of distributions (weak limit).



Physical meaning

Impulse function

- The behavior of $\delta(x)$ is (on the limit) equivalent to an infinitesimally short impulse.
- In physics and engineering, this generalized function is used to model idealizations such as charge point charges, point masses or any other physical quantity concentrated at a zero-extension point.
- Thus, it is said that the support of $\delta(x)$ is a single point $x = 0$.
- The units of $\delta(x)$ are the inverse units of the differential dx.

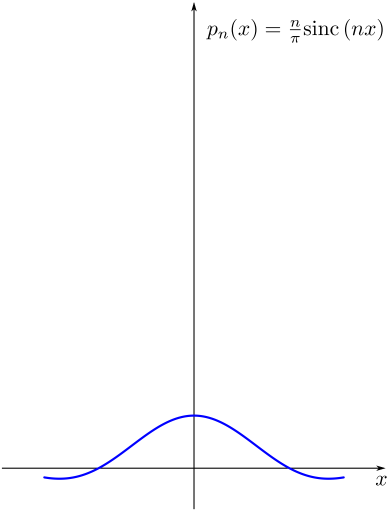
Dirac delta as a limit of a sequence of functions - II

- In addition to pulse function sequences, it is possible to find other sequences of functions $p_1, p_2, \ldots$ such that their limit (in the sense of distributions) is also the Dirac Delta.
- Some useful examples are the sinc and Gaussian functions:

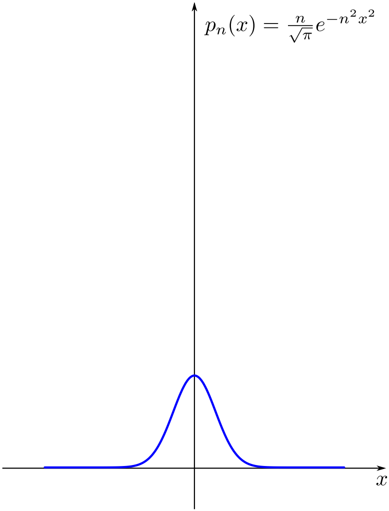
$$\boxed{p_n(x) = n\Pi(nx) \qquad p_n(x) = \frac{n}{\pi}\text{sinc}\,(nx) \qquad p_n(x) = \frac{n}{\sqrt{\pi}}e^{-n^2x^2}}$$

$$\boxed{\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)p_n(x)dx = f(0).}$$

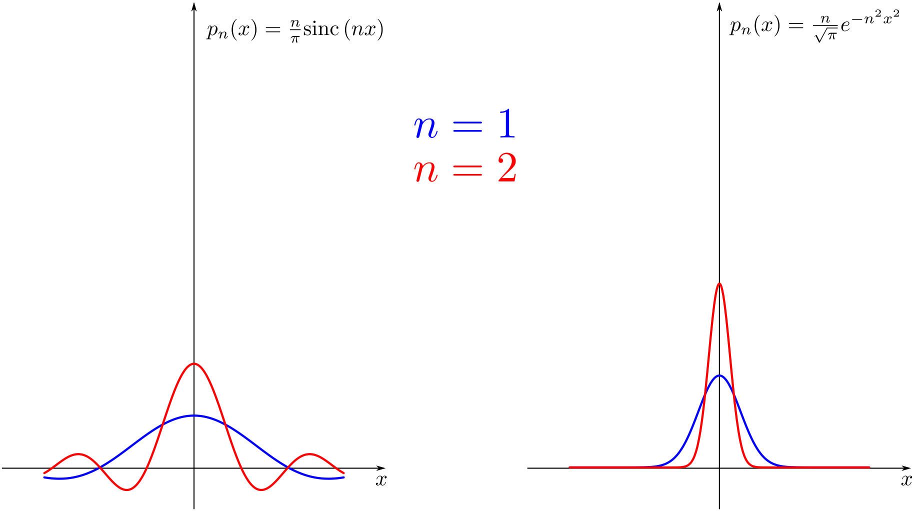
Dirac delta as a limit of a sequence of functions - III



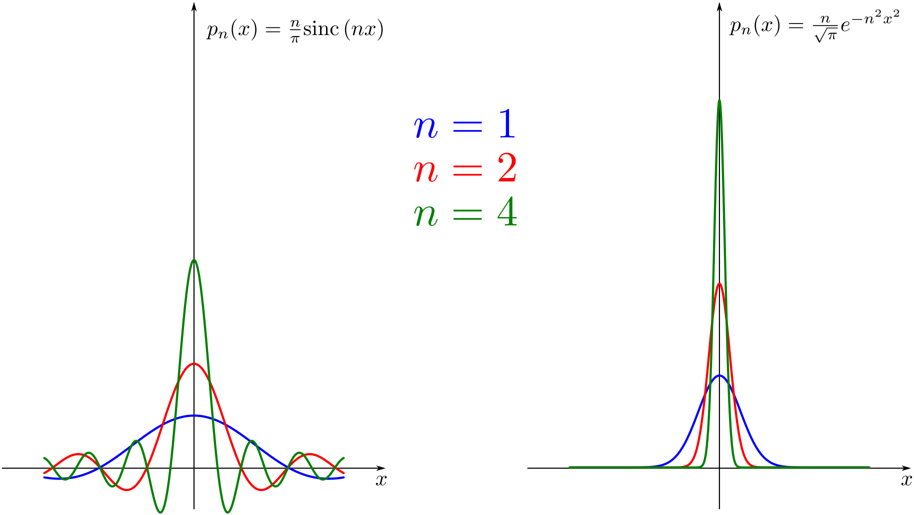
$$n = 1$$



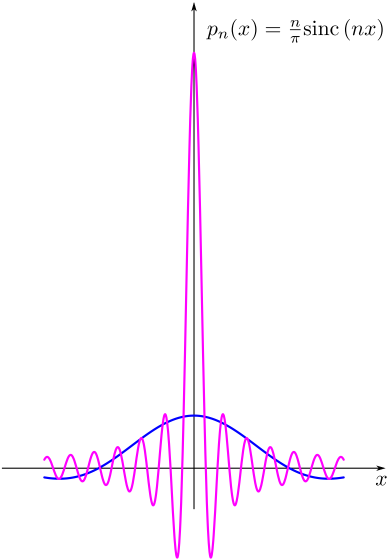
Dirac delta as a limit of a sequence of functions - III



Dirac delta as a limit of a sequence of functions - III



Dirac delta as a limit of a sequence of functions - III

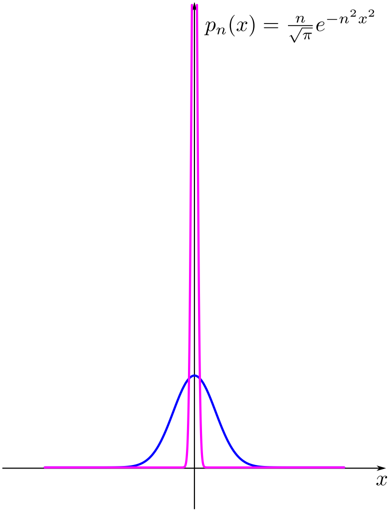


$n = 1$

$n = 2$

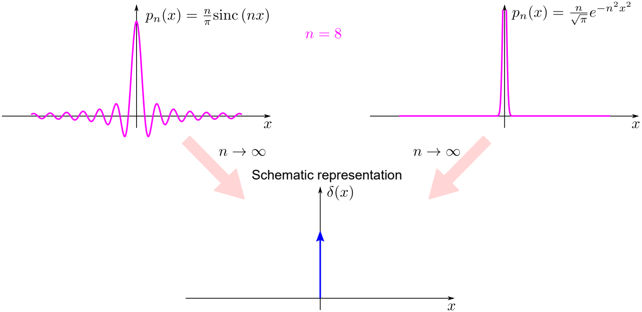
$n = 4$

$n = 8$



Schematic representation

- Due to its physical impulse-like behavior, the Dirac Delta is usually represented by an arrow.
- The height of the arrow is meant to specify the value of any multiplicative constant.



Problems on Basic operations

Exercise

Compute the following operations of the generalized function $\delta(x)$. In addition, sketch the involved distribution.

1. $\int_{-\infty}^{\infty} f(x)\delta(x)dx$
2. $\int_{-\infty}^{\infty} f(x)\delta(-x)dx$
3. $\int_{-\infty}^{\infty} f(x)\delta(x-2)dx$
4. $\int_{-\infty}^{\infty} f(x)\delta(2-x)dx$
5. $\int_{-\infty}^{\infty} f(x)\delta(2x-1)dx$
6. $\int_{-\infty}^{\infty} f(x)\delta(-2x-1)dx$

Problems on Basic operations

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5. $\int_{-\infty}^{\infty} f(x)\delta(2x-1)dx$
6. $\int_{-\infty}^{\infty} f(x)\delta(-2x-1)dx$

Solution

1. $f(0)$
2. $f(0)$
3. $f(2)$
4. $f(2)$
5. $\frac{1}{2}f(\frac{1}{2})$
6. $\frac{1}{2}f(-\frac{1}{2})$

Problems on Basic Operations

Exercise

A function is even if $f(x) = f(-x)$. Show that the generalized function $\delta(x)$ is equivalent, in the sense of distributions, to $\delta(-x)$:

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = \int_{-\infty}^{\infty} f(x)\delta(-x)dx$$

For this reason, by analogy with conventional functions, it is said that the generalized function $\delta(x)$ is an even distribution.

Solution

By means of a simple change of variable

$$\int_{-\infty}^{\infty} \varphi(x)\delta(-x)dx = -\int_{\infty}^{-\infty} \varphi(-y)\delta(y)dy = \int_{-\infty}^{\infty} \varphi(-y)\delta(y)dy = \varphi(0)$$

Outline

Relationship with the unit step function

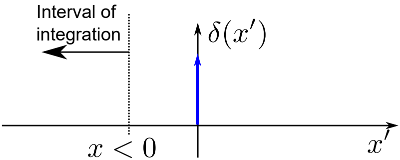
- Applying the properties of the Dirac Delta, it is simple to show that:

$$\boxed{\int_{-\infty}^{x} \delta(x')dx' = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} = u(x)}$$

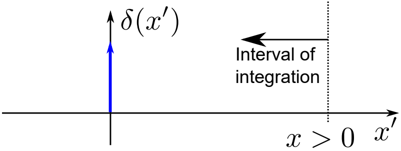
- For this reason, it is said that the unit step function is the primitive of the Dirac Delta.
- Consequently, the derivative (in the sense of distributions) of the Heaviside function is:

$$\boxed{u'(x) = \delta(x)}$$

Relationship with the unit step function



$$\int_{-\infty}^{x} \delta(x')dx' = 0$$



$$\int_{-\infty}^{x} \delta(x')dx' = 1$$

$$\boxed{\int_{-\infty}^{x} \delta(x')dx' = u(x)}$$

Outline

Derivative of discontinuous functions

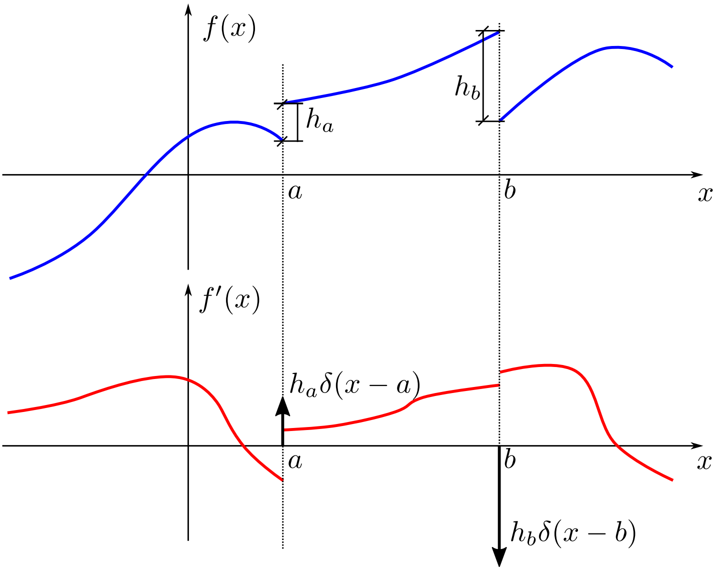
- It is possible to show that the previous result, which relates the derivative of the Heaviside function and the Dirac Delta, can be generalized to a discontinuous function.

Derivative of a discontinuous function

- A well-behaved function with finite jump discontinuities has a derivative at the discontinuity point given by a Dirac Delta, whose amplitude is equal to the jump at the discontinuity point.
- This kind of derivative is usually known as weak derivative, distributional derivative or derivative in the sense of distributions.

(See proof in book by C. Gasquet)

Derivative of discontinuous functions



Derivative of discontinuous functions

Exercise

Compute the weak derivative of $f(x) = |x|$

Solution

$f'(x) = \mathrm{sign}(x)$

Derivative of the Dirac Delta

- In some fields of physics and engineering, in addition to the mathematical object $\delta(x)$, it is also usual to consider another object, whose behavior is equivalent to the corresponding derivative and it is represented by $\delta'(x)$.
- In order to provide an intuitive (but loose) proof, let consider the sequence $p_\Delta(x) = \frac{1}{\Delta}\Pi(\frac{x}{\Delta})$, which (weakly) converges to $\delta(x)$ as $\Delta \to 0$.
- The derivative of these functions is:

$$p'_\Delta(x) = \frac{1}{\Delta}\left(\delta\left(x+\frac{\Delta}{2}\right) - \delta\left(x-\frac{\Delta}{2}\right)\right)$$

- In an analogous way to the study of $\delta(x)$, the limit of this sequence under the integral sign is studied:

$$\lim_{\Delta\to 0}\int_{-\infty}^{\infty} f(x)p'_\Delta(x)dx = \lim_{\Delta\to 0}\frac{1}{\Delta}\left(\int_{-\infty}^{\infty} f(x)\delta\left(x+\frac{\Delta}{2}\right)dx - \int_{-\infty}^{\infty} f(x)\delta\left(x-\frac{\Delta}{2}\right)dx\right)$$

$$= \lim_{\Delta\to 0}\frac{1}{\Delta}\left(f\left(-\frac{\Delta}{2}\right) - f\left(\frac{\Delta}{2}\right)\right) = -\lim_{\Delta\to 0}\frac{f(-\Delta)-f(\Delta)}{2\Delta} = -f'(0)$$

Derivative of the Dirac Delta

- According to the previous result, a definition analogous to the Dirac Delta can be considered:

Derivative of the Dirac Delta

The derivative of the Dirac Delta is defined by its behavior under the integral sign as:

$$\boxed{\int_{-\infty}^{\infty} f(x)\delta'(x-x_0)dx = -f'(x_0)}$$

- This kind of distribution, at it has been previously shown, has all its energy at a single point and, therefore, it is also considered as an impulsive function.
- Moreover, the previous definition can be generalized to define higher order derivatives:

$$\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x)dx = (-1)^n f^{(n)}(0)$$

Derivative in the sense of distributions

- It is interesting to observe that this notation is compatible with conventional integral by parts

$$\int_{-\infty}^{\infty} f(x)g'(x)dx = \underbrace{f(x)g(x)|_{-\infty}^{\infty}}_{0} - \int_{-\infty}^{\infty} f'(x)g(x)dx = -\int_{-\infty}^{\infty} f'(x)g(x)dx$$

  where $g(x) = \delta(x)$, which vanished at it approaches $\pm\infty$ due to its bounded support.

- The generalization of the derivative concept can be performed by analogy with the results from the integration by parts, when the derivative is under the integral sign. In this case, it is required that either $g(x)$ has a bounded support or $f(x)$ vanishes as approaches $\pm\infty$.
  - In addition, $f(x)$ must be differentiable (see further details in the appendix about Theory of Distributions).
- In these cases, the weak derivative, distributional derivative or derivative in the sense of distributions of a function (or distribution) $D(x)$ is defined by:

$$\boxed{\int_{-\infty}^{\infty} f(x)D'(x)dx = -\int_{-\infty}^{\infty} f'(x)D(x)dx}$$

Convolution of distributions - I

- The convolution of distributions exhibits several problems. For instance, in general, it does not satisfy the associative property.
- However, convolution involving impulse distributions (e. g., $\delta(x)$,$\delta'(x)$, etc.) can be defined in in an analogous way to conventional convolutions as it can be show that it does not involve inconsistencies due to the properties of bounded support distributions (see Appendix).
- Hence, it is safe to use conventional operations and definitions:

$$f(x) * \delta(x - x_0) = \int_{-\infty}^{\infty} f(x - x')\delta(x' - x_0)dx' = f(x - x_0)$$

$$f(x) * \delta'(x - x_0) = \int_{-\infty}^{\infty} f(x - x')\delta'(x' - x_0)dx' =$$

$$\int_{-\infty}^{\infty} f(x - y - x_0)\delta'(y)dy =$$

$$-\int_{-\infty}^{\infty} \frac{d(x - y - x_0)}{dy} \left.\frac{df(u)}{du}\right|_{u = x - y - x_0} \delta(y)dy = f'(x - x_0)$$

where the chain rule has been considered.

Convolution of distributions - II

Convolution of distributions

$$\boxed{f(x) * \delta(x - x_0) = f(x - x_0)}$$

$$\boxed{f(x) * \delta'(x - x_0) = f'(x - x_0)}$$

$$\boxed{f(x) * \delta^{(n)}(x - x_0) = f^{(n)}(x - x_0)}$$

These operations are also valid if $f(x)$ is an impulse distribution.

Convolution of distributions- III

Exercise

Compute the following operations

1. $f(x) * \delta(2x-1)$
2. $\delta(x-1) * \delta(x+5)$
3. $\delta(3x-1) * \delta(-2x+1)$
4. $f(3x+1) * \delta(-3x+1)$

Convolution of distributions- IV

Solution

1. $f(x)*\delta(2x-1)=\int_{-\infty}^{\infty} f(x-x')\delta(2x'-1)dx'=\frac{1}{2}\int_{-\infty}^{\infty} f(x-x')\delta(x'-\frac{1}{2})dx'=\frac{1}{2}f(x-\frac{1}{2})$
2. $\delta(x-1)*\delta(x+5)=\delta(x+4)$
3. The associative property must be satisfied for every function $f(x)$:

$$f(x)*(\delta(3x-1)*\delta(-2x+1))=(f(x)*\delta(3x-1))*\delta(-2x+1)$$
$$=\frac{1}{3}f(x-\frac{1}{3})*\delta(-2x+1)=\frac{1}{6}f(x+\frac{1}{6})=f(x)*\left(\frac{1}{6}\delta(x+\frac{1}{6})\right)$$

hence:

$$\delta(3x-1)*\delta(-2x+1)=\frac{1}{6}\delta(x+\frac{1}{6})$$

4. $f(3x+1)*\delta(-3x+1)=\int_{-\infty}^{\infty} f\left(3\left(x-x'\right)+1\right)\delta(-3x'+1)dx'=\frac{1}{3}\int_{-\infty}^{\infty} f(3x-3x'+1)\delta(x'-\frac{1}{3})dx'=\frac{1}{3}f(3x)$

A relevant property of distributions

- When working with distributions, some equations can admit solutions that are more general.
- In this context, is very relevant to know that the equality

$$xD(x) = C$$

has the general solution

$$D(x) = A\delta(x) + \text{p.v.}\frac{1}{x}C$$

where A is a constant to be determined and the distribution p.v.$\frac{1}{x}$, is defined under the integral sign as:

$$⨍_{-\infty}^{\infty} \varphi(x)\frac{1}{x}dx$$

where $⨍$ represents the Cauchy principal value.

- However, in most of engineering texts, the principal value is implicitly supposed and, therefore, the previous solution is written as:

$$\boxed{xD(x) = C \Leftrightarrow D(x) = A\delta(x) + \frac{1}{x}C}$$

Product of distributions

- The product of distributions is usually not defined.
  - The reason is that it does not satisfy some conventional properties such as the associative property. For example:

$$(\delta(x) \times x) \times \text{p.v.}\frac{1}{x} = 0 \times \text{p.v.}\frac{1}{x} = 0$$

$$\text{but} \qquad \delta(x) \times \left(x \times \text{p.v.}\frac{1}{x}\right) = \delta(x) \times 1 = \delta(x)$$

- Nevertheless the product of distributions is well-defined when their support is disjoint.
  - For example, $\delta(x - x_0)\delta(x - x_1) = 0$, if $x_0 \neq x_1$.
  - Nevertheless, the product of two Dirac deltas at the same position, e.g., $\delta(x)\delta(x)$ is not defined.

Outline

Summary of identities of the Dirac Delta

Next, the main properties of the Dirac Delta are summarized using physics/engineering notation:

$f(x)\delta(x-x_0) = f(x_0)\delta(x-x_0)$	$f(x)\delta'(x-x_0) =$ $f(x_0)\delta'(x-x_0) - f'(x_0)\delta(x-x_0)$
$x\delta(x) = 0$	$x\delta'(x) = -\delta(x)$
$f(x) * \delta(x-x_0) = f(x-x_0)$	$f(x) * \delta'(x-x_0) = f'(x-x_0)$
$\delta(ax-x_0) = \frac{1}{\|a\|}\delta(x-\frac{x_0}{a})$	$\delta'(ax-x_0) = \frac{1}{\|a\|}\frac{1}{a}\delta'(x-\frac{x_0}{a})$
$\delta(x) = \delta(-x)$	$\delta'(x) = -\delta'(-x)$