Relevant CV signals

Chapter 4

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Outline I

Outline

Introduction

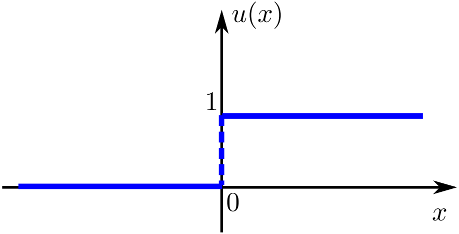
- In this part (chapters 4, 5, 6 and 7), we will focus only on continuous-variable (CV) signals.
- In this chapter, some of the most relevant CV signals for engineering will be introduced.
- These signals will be classified into aperiodic $S(\mathbb{R})$ or periodic $P(X_0)$ signals.

Outline

Unit step function or Heaviside step function

- The unit step or Heaviside step function is defined as follows:

$$\boxed{u(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}}$$



- Note that the function is intentionally not defined at $x = 0$.
- It satisfies the following property:

$$f(x) * u(x) = \int_{-\infty}^{x} f(x')dx'$$

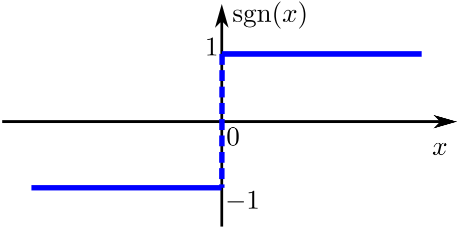
i.e., the convolution with these signals results in an integrator.

- Other authors (e. g., Oppenheim) use a different notation $\Gamma(x) = u(x)$.

Sign function

- The sign function is defined as follows:

$$\boxed{\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}}$$



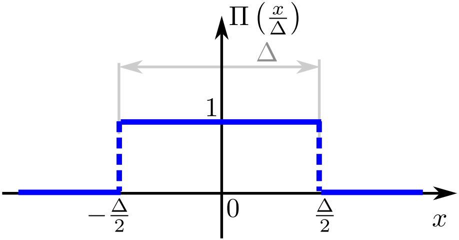
- Note that this function has not been intentionally defined at $x = 0$.
- It meets the following properties:

$$\begin{aligned} \operatorname{sgn}(x) &= 2u(x) - 1 = u(x) - u(-x) \\ \operatorname{sgn}(x) &= \frac{x}{|x|}, \text{ if } x \neq 0 \end{aligned}$$

Unit pulse

- The unit pulse function is defined as follows:

$$\boxed{\Pi\left(\frac{x}{\Delta}\right)=\begin{cases}1 & |x|<\frac{\Delta}{2}\\ 0 & |x|>\frac{\Delta}{2}\end{cases}}$$



- Note that this signal has not been intentionally defined at $x=\pm\frac{\Delta}{2}$.
- It satisfies the following property:

$$f(x)*\Pi\left(\frac{x}{\Delta}\right)=\int_{x-\frac{\Delta}{2}}^{x+\frac{\Delta}{2}}f(x')dx'$$

i.e., the convolution provides the average value of the signal.

Unit pulse

- Other authors (e. g., Emilio) prefer the notation $P_1(x) = \Pi(x)$.

Exercise

Express $P_\Delta(x)$ in terms of the function $\Pi(x)$

Exercise

Express $\Pi\left(\frac{x}{\Delta}\right)$ in terms of the function $u(x)$

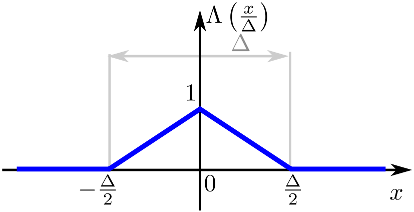
Exercise

Determine and sketch $\Pi\left(\frac{x}{A}\right) * \Pi\left(\frac{x}{B}\right)$

Triangle function

- The triangle function of width Δ is defined as follows:

$$\boxed{\Lambda\left(\frac{x}{\Delta}\right)=\begin{cases}1-\frac{2}{\Delta}\left|x\right| & \left|x\right|\leq\frac{\Delta}{2}\\ 0 & \text{otherwise}\end{cases}}$$



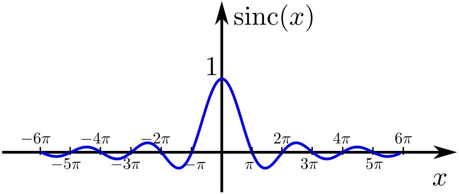
- It is related to the pulse function by means of the following convolution:

$$\Pi\left(\frac{x}{\Delta}\right)*\Pi\left(\frac{x}{\Delta}\right)=\Delta\Lambda\left(\frac{x}{2\Delta}\right)$$

Sinc or cardinal sine function

- The sinc or cardinal sine function is defined as follows:

$$\boxed{\mathrm{sinc}(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}}$$

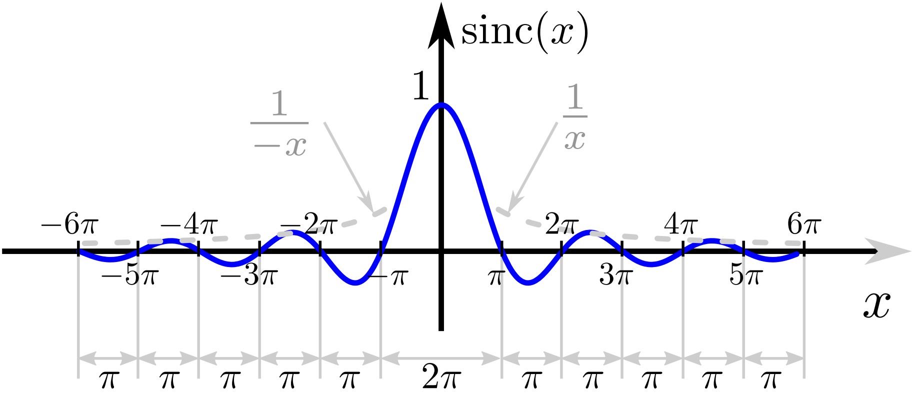


- In many works, it is differently defined as $\cancel{\frac{\sin(\pi x)}{\pi x}}$, which is sometimes referred to as normalized sinc function.
- It is worthwhile to mention that the zeros are at integer multiples of π:

$$\mathrm{sinc}(x_0) = 0 \Leftrightarrow x_0 = n\pi,\ n \in \mathbb{Z} \setminus \{0\}$$

Sinc or cardinal sine function

Although the meaning of the sinc function is not clear at this stage of the course, it will play a key role here and, moreover, in the rest of the major.



Outline

Sinusoidal signals

- The most widespread periodic signals are the sinusoidal ones, which can be expressed in terms of either the sine function or the cosine function.

$$f_0(x) = A\sin(\xi_0 x + \varphi_0) \text{ or } f_0(x) = A\cos(\xi_0 x + \varphi_0)$$

- A specifies the amplitude of the signals.
- In these signals, the term multiplying the independent variable, ξ_0, is referred to as the angular frequency.
  - The fundamental period of the signal is given by $X_0 = \frac{2\pi}{\xi_0}$.
  - The frequency of the signal is given by $\frac{\xi_0}{2\pi} = \frac{1}{X_0}$.

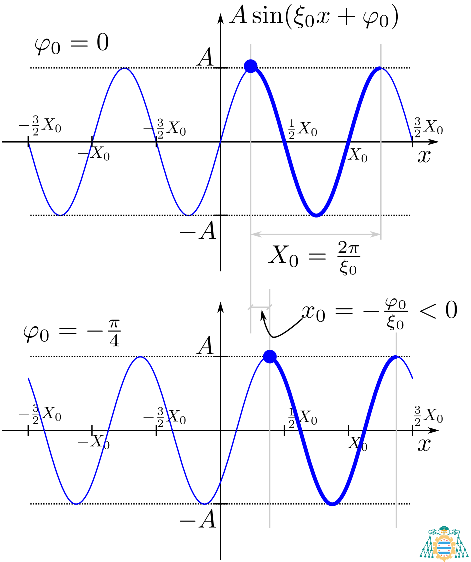
Sinusoidal signals

- The phase of the signal φ_0 manages the shift of the independent variable:

$$f_0(x) = A\sin\left(\xi_0 x + \varphi_0\right) = A\sin\left(\xi_0\left(x + \frac{\varphi_0}{\xi_0}\right)\right) = A\sin\left(\xi_0\left(x - x_0\right)\right)$$

where $x_0 = -\frac{\varphi_0}{\xi_0}$.

- $\varphi_0 > 0$ shift to the left (advanced signal)
- $\varphi_0 < 0$ shift to the right (delayed signal)



Sinusoidal signals

- Depending on the domain of the independent variable (time or space), different "standard" notations can be used

	Time		
	Usual notation	Name	Units
Period (X_0)	T_0	Period	s
Angular frequency (ξ_0)	ω_0	Angular frequency	rad/s
Frequency	f_0	Frequency	Hz=s^{-1}

	Space		
	Usual notation	Name	Units
Period (X_0)	λ	Wavelength	m
Angular frequency(ξ_0)	k_0, β_0	Wavenumber	rad/m
Frequency	-	-	m^{-1}

Sinusoidal signals

Exercise

Determine the inner product of the sine and cosine signals

$$\langle \cos(\xi_0 x), \sin(\xi_0 x) \rangle$$

Exercise

Determine the energy per period of the previous signals
$E_0[f_0(x)] = \int_{\langle X_0 \rangle} |f(x)|^2 \, dx$

Exercise

Find the required conditions for $\xi_{0,1}$ and $\xi_{0,2}$ so that the following inner product is zero:

$$\langle \sin(\xi_{0,1} x), \sin(\xi_{0,2} x) \rangle = 0$$

Exponential with imaginary exponent

- Exponentials functions with imaginary exponent are as follows:

$$f_0(x) = e^{j(\xi_0 x + \varphi_0)}$$

- These exponential can be decomposed into a linear combination of sinusoidal functions by means of the Euler's formula:

$$\boxed{f_0(x) = e^{j(\xi_0 x + \varphi_0)} = \cos(\xi_0 x + \varphi_0) + j \sin(\xi_0 x + \varphi_0)}$$

- Note that $e^{j(\xi_0 x + \varphi_0)} \in P(X_0)$ where $X_0 = \frac{2\pi}{\xi_0}$.
- In addition, a decomposition in terms of modulus and argument (also known as phase) is also convenient in many cases:

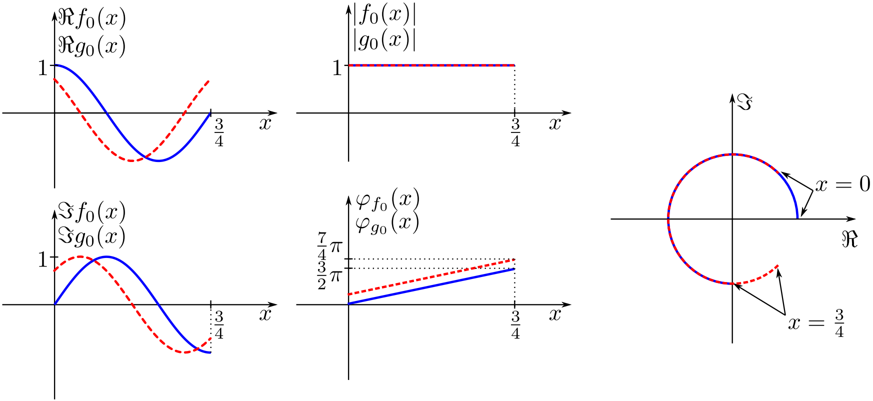
$$\left| e^{j(\xi_0 x + \varphi_0)} \right| = 1 \quad \varphi_{f_0} = \xi_0 x + \varphi_0$$

$$\Re\left\{ e^{j(\xi_0 x + \varphi_0)} \right\} = \cos(\xi_0 x + \varphi_0) \quad \Im\left\{ e^{j(\xi_0 x + \varphi_0)} \right\} = \sin(\xi_0 x + \varphi_0)$$

Exponential with imaginary exponent

$f_0(x) = e^{j2\pi x}$
$g_0(x) = e^{j(2\pi x + \frac{\pi}{4})}$

interval $0 < x < \frac{3}{4}$



Outline

Complex exponential

- Complex exponentials will be one of the most important functions in this course:

$$f(x) = Fe^{s_0 x},\ s_0, F \in \mathbb{C}$$

- To understand their behavior, it is convenient to decompose them into real/imaginary or modulus/argument parts:

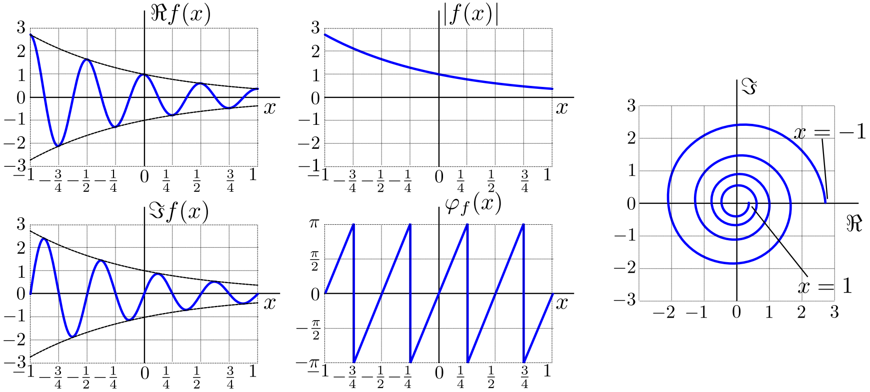
$$\begin{matrix} s_0 = s_0' + js_0'' \\ F = |F|e^{j\varphi_F} \end{matrix} \Rightarrow f(x) = \underbrace{|F|e^{s_0' x}}_{\in S(\mathbb{R})} \underbrace{e^{j\left(s_0'' x + \varphi_F\right)}}_{\in P(X_0)}$$

The first term corresponds to a real exponential $|F|e^{s_0' x}$ whereas the second one corresponds to an exponential with imaginary exponent $e^{j\left(s_0'' x + \varphi_F\right)}$

- Depending on the real part of the exponent, the signal can be periodic ($s_0' = 0$) or aperiodic ($s_0' \neq 0$).

Complex exponential

$$f(x) = e^{(-1+j4\pi)x} \quad \text{interval } -1 < x < 1$$



Complex exponential

$$f(x) = e^{(1+j4\pi)x} \quad \text{interval } -1 < x < 1$$