Operator Spaces: Systems

Chapter 3

Jaime Laviada

Área de Teoría de la Señal y Comunicaciones
Universidad de Oviedo

Outline I

1. Introduction

2. Mapping between signal spaces
   - Functionals
   - Bilinear Form
   - Operators
   - Summary of mappings between signal spaces

Outline II

3. Systems
   - Examples of systems
   - Basic system properties
   - Linearity
   - Invariance
   - Memory
   - Invertibility
   - Causality
   - Stability
   - Interconnections of systems
4. The convolution operator
   - Convolution on aperiodic signals
   - Examples
   - Convolution on periodic signals

Outline III

- Properties of convolution

Outline IV

5. The Correlation Operator
   - Crosscorrelation
   - Example

Outline

1 Introduction

2 Mapping between signal spaces

3 Systems

4 The convolution operator

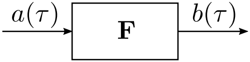
5 The Correlation Operator

Introduction

What is a system?

- A system is a process in which input signals are transformed resulting in other signals (output signals).
- This process can be view as a **black box** transforming input signals into output signals.
- From a mathematical point of view, they are represented by means of operators.

In this course, we will focus on systems with only one input signal and only one output signal.



Outline

1 Introduction

2 Mapping between signal spaces
- Functionals
- Bilinear Form
- Operators
- Summary of mappings between signal spaces

3 Systems

4 The convolution operator

5 The Correlation Operator

Mapping between signal spaces

- Operators are a specific kind of mapping between sets of elements (spaces).
- In addition to operators, there are also several additional mappings between vector spaces
    - Functionals
    - Bilinear forms
    - Operators
- Abstract algebraic concepts will be skipped in this chapter in order to focus on the basic concepts.

Functionals

- The term functional refers to a mapping from a vector space into its field of scalars:

$$\begin{aligned} F : \mathcal{V}_{\mathcal{K}} &\to \mathcal{K} \\ \vec{a} &\mapsto \alpha = F[\vec{a}] \end{aligned}$$

- In the case of signal spaces, it maps every function in the space into a (real or complex) number.
- A functional is linear if:

$$F\left[\alpha\vec{a} + \beta\vec{b}\right] = \alpha F\left[\vec{a}\right] + \beta F\left[\vec{b}\right], \text{ where } \alpha, \beta \in \mathcal{K},\ \vec{a}, \vec{b} \in \mathcal{V}_{\mathcal{K}}$$

- **Example**: the norm defined on a given vector space, the energy of a signal or the distributions (see chapter 5).

Bilinear and multilinear forms

- A bilinear form is a bilinear map combining elements from two vector spaces to yield a scalar:

$$F : \mathcal{V}_{\mathcal{K}_1} \times \mathcal{W}_{\mathcal{K}_2} \to \tilde{\mathcal{K}}$$
$$\left(\vec{a}, \vec{b}\right) \mapsto \alpha = F\left[\vec{a}, \vec{b}\right]$$

- In addition, they must satisfy a number of properties regarding linearity with respect to each argument.
  - These properties can be found in the book by Emilio Gago.
- In general, the domains and the codomain are the same vector space $\mathcal{V}_{\mathcal{K}_1} = \mathcal{W}_{\mathcal{K}_2} = \mathcal{V}_{\mathcal{K}}$ and $\tilde{\mathcal{K}} = \mathcal{K}$.
- A multilinear form is similar but the number of domains is larger than 2.
- **Example**: the inner product

Operators

- An operator is a mapping between two vector spaces:

$$\begin{aligned}\mathbf{F} : \mathcal{V}_{\mathcal{K}_1} &\to \mathcal{W}_{\mathcal{K}_2} \\ \vec{a} &\mapsto \vec{b} = \mathbf{F}\left[\vec{a}\right]\end{aligned}$$

- In addition, bivariate and multivariate operators can also be defined, involving two or more inputs.
  - They will not be studied in this course.
- In many cases, both vector spaces are the same ones $\mathcal{V}_{\mathcal{K}_1} = \mathcal{W}_{\mathcal{K}_2} = \mathcal{V}_{\mathcal{K}}$.
  - However, cases not meeting the previous condition will also be considered in this course.
- An operator is said to be linear if

$$\mathbf{F}\left(\alpha\vec{a}+\beta\vec{b}\right)=\alpha\mathbf{F}(\vec{a})+\beta\mathbf{F}\left(\vec{b}\right), \text{ where } \vec{a},\vec{b}\in\mathcal{V}_{\mathcal{K}}, \mathbf{F}(\vec{a}),\mathbf{F}\left(\vec{b}\right)\in\mathcal{W}_{\mathcal{K}}, \alpha,\beta\in\mathcal{K}$$

where it is supposed that the same field of scalars is used in both vector spaces.

Operator examples

- Let's see some examples:

$$b(\tau) = \mathbf{F}\left(a\left(\tau\right)\right) = a\left(\tau\right)^2 \qquad g(x) = \mathbf{F}\left(f\left(x\right)\right) = \left|f\left(x\right)\right|$$

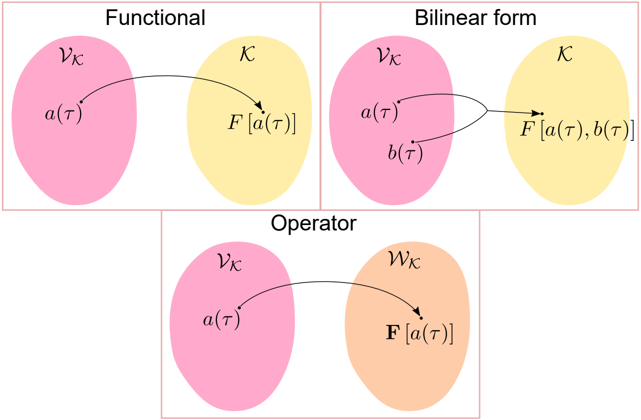
$$g(x) = \mathbf{F}\left(f\left(x\right)\right) = \int_{-\infty}^{x} f\left(x'\right) dx' \qquad g(x) = \mathbf{F}\left(f\left(x\right)\right) = f\left(2-3x\right)$$

$$y(n) = \mathbf{F}\left(x\left(n\right)\right) = \begin{cases} 2x(n) & n \leq 0 \\ 3x(n) & n > 0 \end{cases} \qquad y(n) = \mathbf{F}\left(x\left(n\right)\right) = x(n) + \sin(2\pi n)$$

- In addition to systems, signal transforms can also be described by means of operators.

Summary of mappings between signal spaces

- Next, the main mappings between signal spaces are summarized in its common forms:



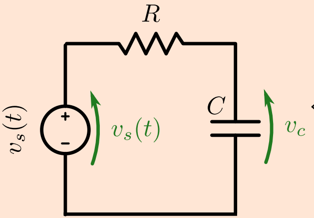
- These signal spaces can be $S(\mathbb{R})$, $P(X_0)$, $D(\mathbb{Z})$, $D(N_0)$, etc.
- The field of scalars $\mathcal{K}$ can be either $\mathbb{R}$ or $\mathbb{C}$.

Outline

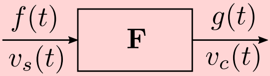
1. Introduction
2. Mapping between signal spaces
3. Systems
   - Examples of systems
   - Basic system properties
   - Linearity
   - Invariance
   - Memory
   - Invertibility
   - Causality
   - Stability
   - Interconnections of systems

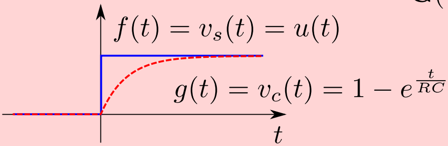
Example - Electric circuit

RC circuit



$$\begin{cases} i(t) = \frac{v_s(t) - v_c(t)}{R} \\ i(t) = C\frac{dv_c(t)}{dt} \end{cases} \Rightarrow RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$





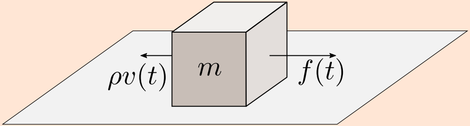
Operator notation

$$\mathbf{G}(\cdot) = RC\frac{d}{dt}(\cdot) + (\cdot) \qquad \mathbf{G}(v_c(t)) = v_s(t)$$

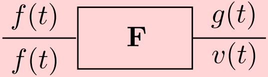
$$v_c(t) = \mathbf{F}(v_s(t)) \text{ where } \mathbf{F} = \mathbf{G}^{-1}$$

Example - Force applied to an object

Acceleration of an object



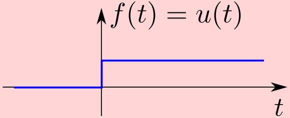
$$\left\{\begin{array}{c} ma(t) = f(t) - \rho v(t) \\ a(t) = \frac{dv(t)}{dt} \end{array}\right. \Rightarrow m\frac{dv(t)}{dt} + \rho v(t) = f(t)$$

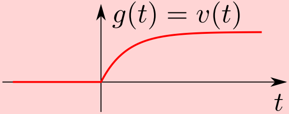


Operator notation

$$\mathbf{G}(\cdot) = m\frac{d}{dt}(\cdot) + \rho(\cdot) \qquad \mathbf{G}(v(t)) = f(t)$$

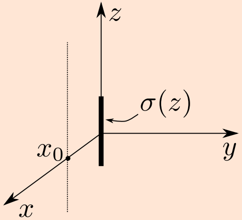
$$v(t) = \mathbf{F}(f(t)) \text{ where } \mathbf{F} = \mathbf{G}^{-1}$$





Example - Electric potential

Electric potential due to linear charge density

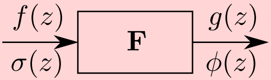


Potential at line x= x_0 and y=0

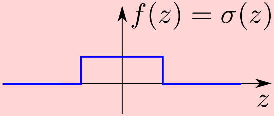
$$\phi(x_0, y=0, z) = \phi(z) = \frac{1}{4\pi\varepsilon}\int \frac{\sigma(z')}{R(z,z')}dz'$$

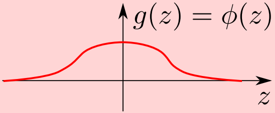
$$R(z,z') = \sqrt{(z-z')^2 + x_0^2}$$

Operator notation



$$\mathbf{F}(\cdot) = \frac{1}{4\pi\varepsilon}\int (\cdot)\frac{1}{R(z,z')}dz' \qquad \phi(z) = \mathbf{F}(\sigma(z))$$





Example - Wealth of a bank account

Wealth of a bank account

$y(n)$, wealth at the beginning of year n
$x(n)$, money saved at the end of year n
r, interest rate

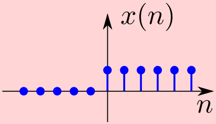
$$y(n+1) = (1+r)y(n) + x(n)$$

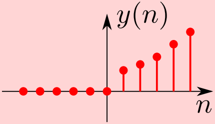
Operator notation

$$x(n) \longrightarrow \boxed{\mathbf{F}} \longrightarrow y(n)$$

$$\mathbf{F}^{-1}(\cdot) = \mathbf{I}(\cdot) - (1+r)\mathbf{D}_{-1}(\cdot)$$

$$x(n) = \mathbf{F}^{-1}(y(n))$$
$$y(n) = \mathbf{F}(x(n))$$





$\mathbf{I}(\cdot)$, identity operator
$\mathbf{D}_m(\cdot)$, right shift operator

Basic system properties

- Next, the basic system properties will be studied.
- It is important to note that these properties have to be met for **all** signals.
- It is enough to find a counterexample to show that system does not satisfy a given property.

Linear systems

- A system is said to be linear if it preserves linear combinations with the same weights:

$$\left.\begin{array}{l} a_1(\tau) \rightarrow b_1(\tau) \\ a_2(\tau) \rightarrow b_2(\tau) \end{array}\right\} \Rightarrow \boxed{\alpha a_1(\tau) + \beta a_2(\tau) \rightarrow \alpha b_1(\tau) + \beta b_2(\tau)}$$

- This property is defined analogously for the operator modeling the system.
- If a system is linear and well-behaved functions are considered, then the system also preserves generalized linear combinations (integrals):

$$\mathbf{F}: \; a(\tau) = \int_\xi \alpha(\xi) e(\tau;\xi) d\xi \rightarrow b(\tau) = \mathbf{F}\left[a(\tau)\right] = \int_\xi \alpha(\xi) \mathbf{F}\left[e(\tau;\xi)\right] d\xi$$

- This property is sometimes referred to as superposition.

Linear systems

Exercise

Determine whether or not the following systems are linear:

1. $g(x) = K\frac{df(x)}{dx}$ $\mathbf{F}[\cdot] = K\frac{d[\cdot]}{dx}$
2. $g(x) = K\left|f(x)\right|$ $\mathbf{F}[\cdot] = K\left|\cdot\right|$
3. $y(n) = x(n) + K$ $\mathbf{F}[\cdot] = \cdot + K$
4. $y(n) = x(n) - x(n-1)$ $\mathbf{F}[\cdot] = \cdot - \mathbf{D}_1[\cdot]$
5. $y(n) = \begin{cases} x(n) & \text{if } x(0) \neq x(1) \\ 0 & \text{otherwise} \end{cases}$

Exercise

Show that any zero-input signal to a linear system results in a zero-output signal:

$$a(\tau) = 0 \rightarrow b(\tau) = 0$$

Shift-invariant systems

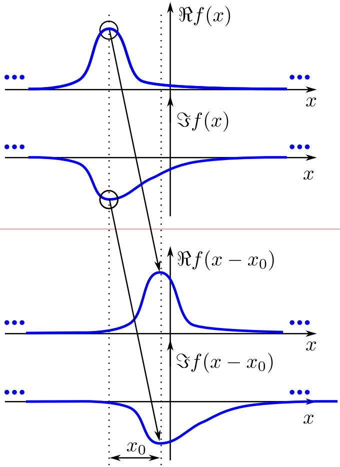
- A system is said to be shift-invariant (or just invariant) if a shift (along the independent variable) in the input signal results in an identical shift in the output signal:

$$a(\tau) \to b(\tau) \Rightarrow \boxed{a(\tau - \tau_0) \to b(\tau - \tau_0)}$$

- Consequently, the system behavior does not depend on the origin of coordinates.
- In the time domain, it means that system properties do not change over time.

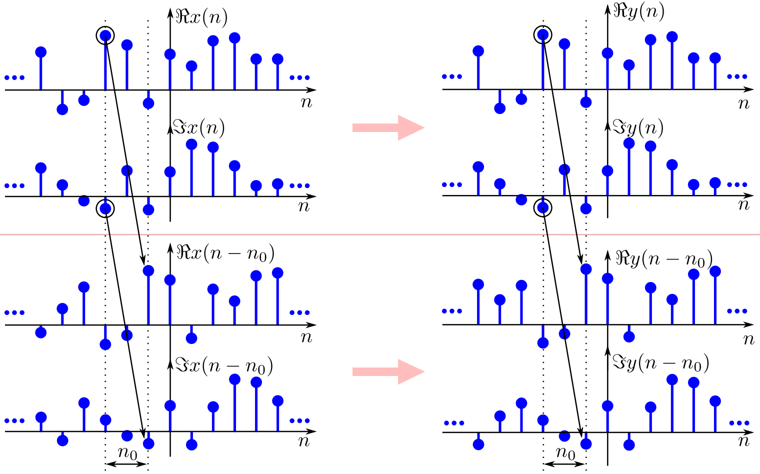
Shift-invariant systems

Invariance - Continuous variable

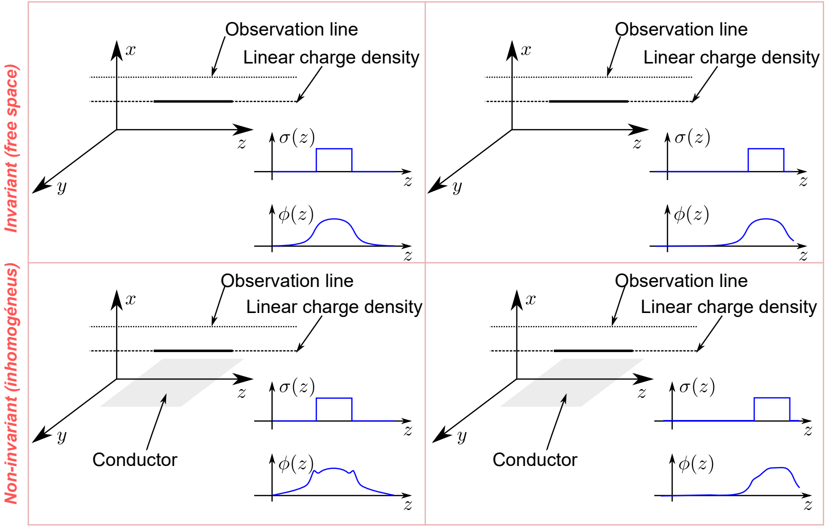


Shift-invariant systems

Invariance - Discrete-variable



Shift-invariant systems



Shift-Invariant systems

Exercise

Determine whether or not the following systems are invariant:

1. $g(t) = \mathbf{F}[f(t)],\ \mathbf{F}[f(t)] = \begin{cases} f(t) & t \geq 0 \\ 0 & t < 0 \end{cases}$
2. $y(n) = x(n) + p(n),\ p(n) \in D(\mathbb{Z})$
3. $y(n) = \sum_{m=-\infty}^{n} x(m)$

Systems with and without memory

- A system is said to be memoryless if its output for each value of the independent variable is dependent only on the input at the same value of the independent variable.

$$\boxed{b(\tau_0) = \mathbf{F}\left[a(\tau_0)\right], \forall \tau_0}$$

in other words, the output at τ_0 depends only on the input value at τ_0.

- For instance, a system modeling the voltage (input) and current (output) at a resistor is a memoryless system:

$$i(t) = \frac{v(t)}{R}$$

- Nevertheless, a capacitor is an example of system with memory, since if the input is taken to be the current and the output is the voltage, then:

$$v(t) = \frac{1}{C} \int_{-\infty}^{t} i(t')dt'$$

and, therefore, the previous values are required to find the voltage at the current time.

Systems with and without memory

Exercise

Determine whether or not each of the following systems is memoryless:

1. $g(x) = f(x)^2,\ \mathbf{F}[\cdot] = (\cdot)^2$
2. $g(x) = f(2x)$
3. $y(n) = \sum_{m=-\infty}^{n} x(m)$
4. $y(n) = x(n) - x(n-1)$

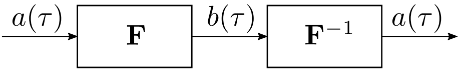
Invertibility

- A system is said to be invertible if there is another system such that when cascaded with the original system yields an output equal to the input of the original system.

$$\boxed{\mathbf{F} \text{ is invertible if } \exists \mathbf{F}^{-1} \text{ such that } \mathbf{F}^{-1}\left[\mathbf{F}\left[a(\tau)\right]\right] = a(\tau), \forall a(\tau)}$$

- The previous definition is equivalent to state that a system is invertible if distinct inputs lead to distinct outputs.

$$\boxed{a_1(\tau) \neq a_2(\tau) \Rightarrow b_1(\tau) \neq b_2(\tau), \forall a_1(\tau), a_2(\tau); \forall \tau}$$



Invertibility

Exercise

Determine whether or not the following systems are invertible:

1. $g(x) = \frac{df(x)}{dx}$
2. $g(x) = f(2x)$
3. $y(n) = x(n)^2$
4. $y(n) = \sum_{m=-\infty}^{n} x(m)$

Causality

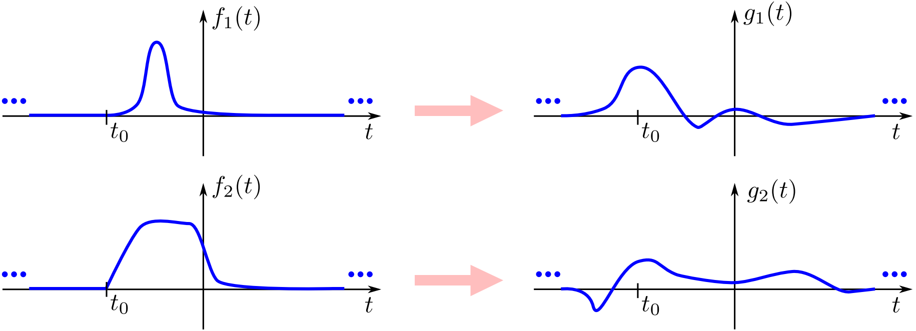
- Causal systems are only interesting when working with time domain signals.
- A system is said to be causal if the output at any τ_0 depends only on values of the input signals at τ_0 or previous values:

$$\boxed{b(\tau_0) = \mathbf{F}\left[a(\tau \leq \tau_0)\right], \forall \tau_0}$$

- Consequently, if two inputs to a causal system are identical up to some point τ_0, then the outputs must also be identical up to this same point.
- Memoryless systems are always causal systems.

Causality

The following system would not be causal since $f_1(t) = f_2(t) = 0, \forall t < t_0$ but $g_1(t) \neq g_2(t), \forall t < t_0$



Causality

Exercise

Determine whether or not the following systems are causal:

1. $g(x) = f(x-1)$
2. $g(x) = f(x+1)$
3. $y(n) = \sum_{m=n-2}^{n+2} x(m)$
4. $y(n) = x(n) + \cos(5.2\pi n)$

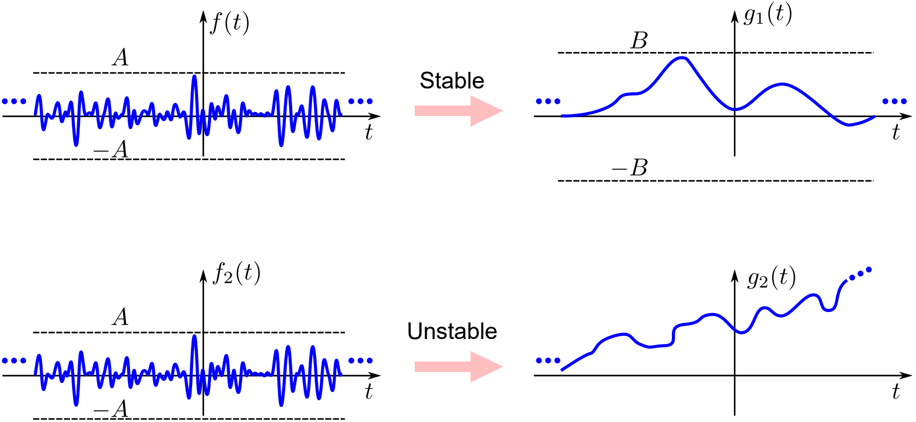
Stability

- A system is said to be stable if any bounded input yields a bounded output:

$$\boxed{\forall a(\tau) \text{ s. t. } |a(\tau)| \leq A, \forall \tau \Rightarrow \exists B \text{ s. t. } |b(\tau)| \leq B, \forall \tau; A, B \in \mathbb{R}^+}$$

- This property is very relevant from an engineering point of view.
  - An unstable system (e.g., an electronic circuit) could produce a signal that could damage other components. Furthermore, this output could happen even due to unwanted small-amplitude signals (e.g., noise).

Stability



Stability

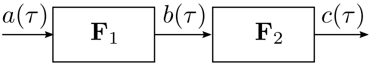
Exercise

Determine whether or not the following systems are stable:

1. $y(n) = nx(n)$
2. The described bank account
3. $g(x) = \int_{-\infty}^{x} f(y)dy$
4. $g(x) = e^{f(x)}$

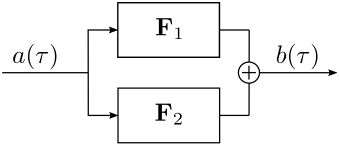
Interconnections of systems

Series (cascade) interconnection



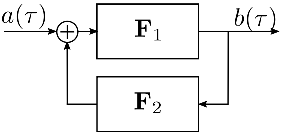
$$\boxed{c(\tau) = \mathbf{F}_2\left[b(\tau)\right] = \mathbf{F}_2\left[\mathbf{F}_1\left[a(\tau)\right]\right]}$$

Parallel interconnection



$$\boxed{b(\tau) = \mathbf{F}_1\left[a(\tau)\right] + \mathbf{F}_2\left[a(\tau)\right]}$$

Feedback interconnection



$$\boxed{b(\tau) = \mathbf{F}_1\left[a(\tau) + \mathbf{F}_2\left[b(\tau)\right]\right]}$$

Outline

1. Introduction
2. Mapping between signal spaces
3. Systems
4. The convolution operator
   - Convolution on aperiodic signals
   - Examples
   - Convolution on periodic signals
   - Properties of convolution
5. The Correlation Operator

Introduction to convolutions

- One of the most relevant operations on signals is the convolution.
- This operation takes two signals, $a(\tau)$ and $b(\tau)$, and produces a new signal:

$$c(\tau) = (a * b)(\tau)$$

however, the following notation is preferred in engineering:

$$c(\tau) = a(\tau) * b(\tau)$$

- Thus, it is a bivariate operator (and, as it will be shown later, bilineal).
- As it will be later illustrated, it is very relevant in the case of linear systems.
- The definition of this operator depends on the considered signal space.

Aperiodic signals

- In the case of signals in $S(\mathbb{R})$, the convolution, which is sometimes referred to as linear convolution, is defined as:

$$\boxed{c(x) = f(x) * g(x) = \int_{-\infty}^{\infty} f(x')g(x-x')dx'}$$

- In the case of discrete-variable signals in $D(\mathbb{Z})$, the convolution is defined in a similar way:

$$\boxed{c(n) = x(n) * y(n) = \sum_{n'=-\infty}^{\infty} x(n')y(n-n')}$$

- In both cases, the convergence of the integral/summation is sometimes tricky. However, this operation will converge for most of the common signals.

Aperiodic signals

Exercise

Given $f(x)$ and $g(x)$, both with bounded support (i.e., finite-length) between $[a, b]$ and $[c, d]$, respective, determine the support of the function $c(x) = f(x) * g(x)$. Repeat this exercise for the discrete-variable case.

Exercise

Show that if the convolution of an aperiodic signal $a(\tau)$ and a periodic signal $b_0(\tau)$ converges, then it is a periodic signal

$$c_0(\tau) = a(\tau) * b_0(\tau)$$

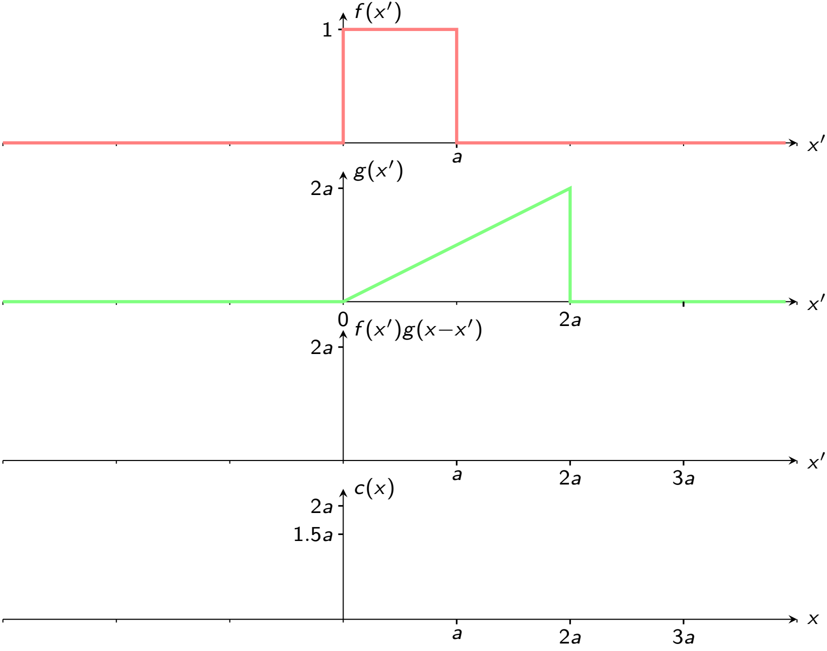
It is recommended to carry out this proof for both cases: CV and DV.

Steps to perform a convolution

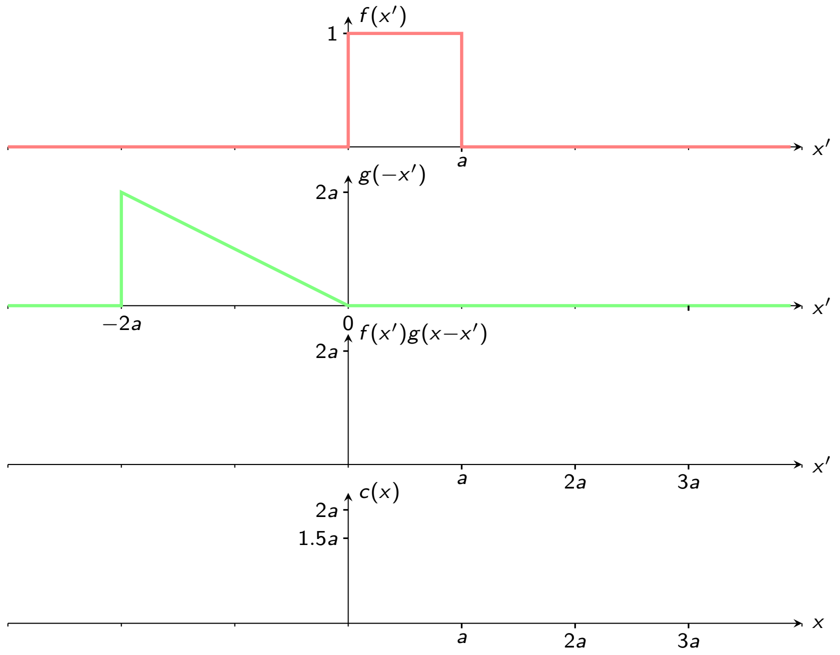
Let us consider that we want to determine the value of a convolution at a given point $x = x_0$, this can be achieved by means of the following steps:

1. Reversing the signal $g(x') \rightarrow g(-x')$
2. Shifting this function a given number of units x_0, $g(-x') \rightarrow g(x_0 - x')$
3. Calculating the multiplication of both functions $f(x')g(x_0 - x')$
4. Integrating the function from the previous step.

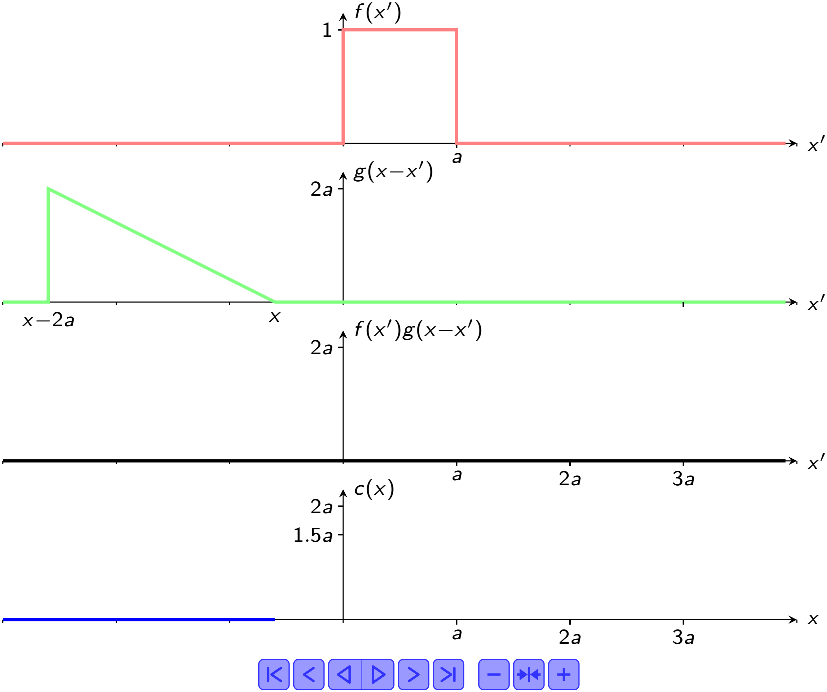
These steps can be repeated to find the value at additional points.
The discrete-variable case can be computed in a similar way as interactively illustrated in the next examples.



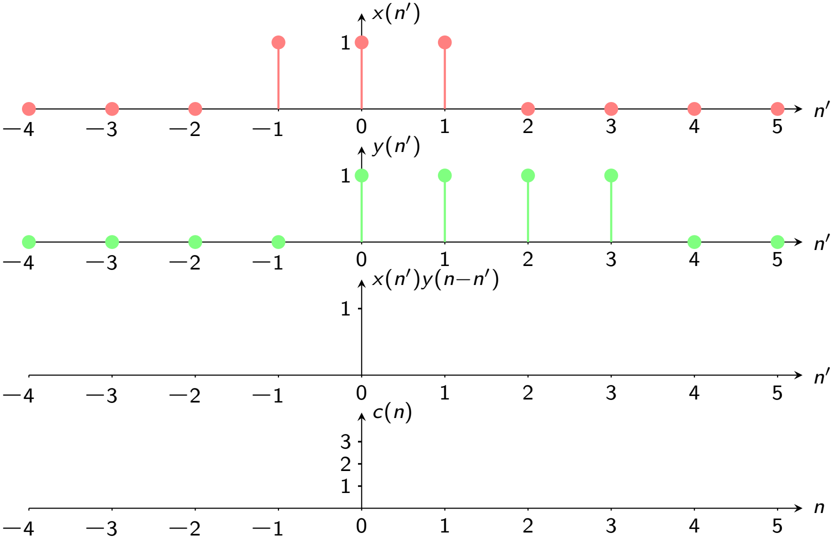
$f(x')$
1
a
x'
$g(x')$
$2a$
0
$2a$
x'
$f(x')g(x-x')$
$2a$
a
$2a$
$3a$
x'
$c(x)$
$2a$
$1.5a$
a
$2a$
$3a$
x



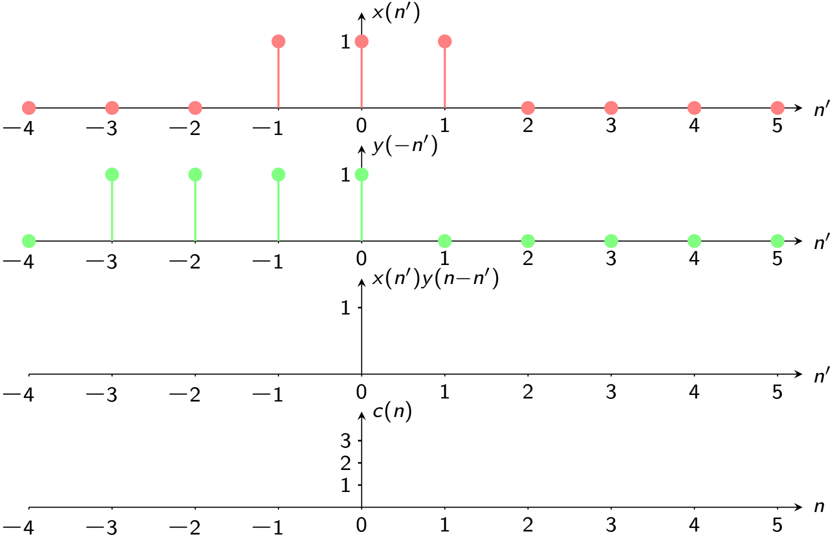
$f(x')$
1
x'
a
$g(-x')$
$2a$
x'
$-2a$
0
$f(x')g(x-x')$
$2a$
x'
a
$2a$
$3a$
$c(x)$
$2a$
$1.5a$
x
a
$2a$
$3a$



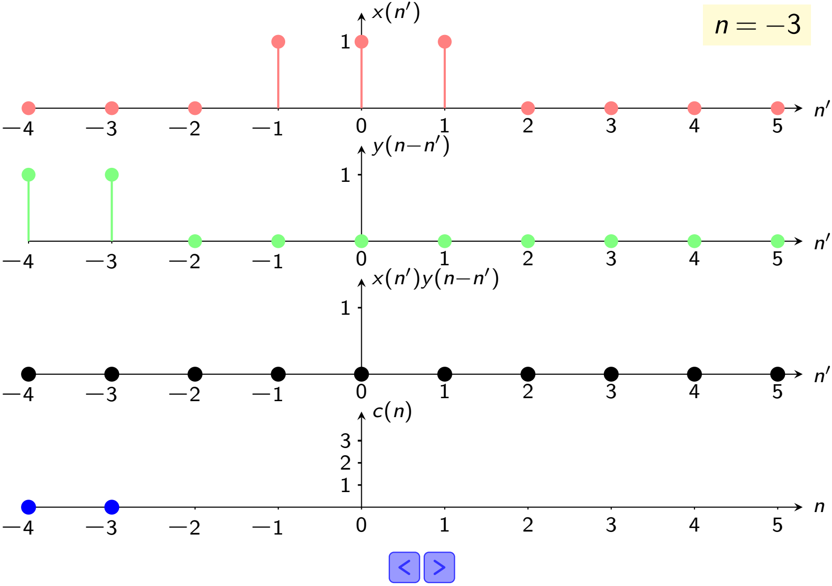
f(x′)
1
x′
a
g(x−x′)
2a
x′
x−2a
x
f(x′)g(x−x′)
2a
x′
a
2a
3a
c(x)
2a
1.5a
x
a
2a
3a



x(n′)
1
n′
−4 −3 −2 −1 0 1 2 3 4 5
y(n′)
1
n′
−4 −3 −2 −1 0 1 2 3 4 5
x(n′)y(n−n′)
1
n′
−4 −3 −2 −1 0 1 2 3 4 5
c(n)
3
2
1
n
−4 −3 −2 −1 0 1 2 3 4 5



x(n′)
1
n′
−4 −3 −2 −1 0 1 2 3 4 5
y(−n′)
1
n′
−4 −3 −2 −1 0 1 2 3 4 5
x(n′)y(n−n′)
1
n′
−4 −3 −2 −1 0 1 2 3 4 5
c(n)
3
2
1
n
−4 −3 −2 −1 0 1 2 3 4 5


$n = -3$
$x(n')$
1
n'
−4 −3 −2 −1 0 1 2 3 4 5
$y(n-n')$
1
n'
−4 −3 −2 −1 0 1 2 3 4 5
$x(n')y(n-n')$
1
n'
−4 −3 −2 −1 0 1 2 3 4 5
$c(n)$
3
2
1
n
−4 −3 −2 −1 0 1 2 3 4 5

Periodic convolution

- If both signals are periodic, then a different definition, referred to as periodic convolution, is usually considered for both cases (CV and DV):

$$\boxed{c_0(x) = f_0(x) * g_0(x) = \int_{\langle X_0 \rangle} f_0(x')g_0(x-x')dx'}$$

$$\boxed{c_0(n) = x_0(n) * y_0(n) = \sum_{n'=\langle N_0 \rangle} x_0(n')y_0(n-n')}$$

in both cases, the chosen interval does not have any impact in the final result as long as it is continuous and with a length equal to the fundamental period.

Periodic convolution

- This convolution is very relevant in the case of transforms of discrete-variable signals.
- In addition, it is closely related to the circular or cyclic convolution, which will be defined when considering those transforms.
- This tool is briefly described next:
  - It is usual to expand a finite-length aperiodic signal by means of a periodic summation:

$$a_{0,p}(\tau) = \sum_{m=-\infty}^{\infty} a(\tau - n\tau_0)$$

    in these cases, as it will be later described, it is necessary to perform "circular convolutions".
  - A circular convolution is just a periodic convolution applied to signals extended by means of periodic summations:

$$a(\tau) \circledast b(\tau) = a_{0,p}(\tau) * b_{0,p}(\tau)$$

Periodic convolution

Exercise

Show that the periodic convolution of two periodic signals is also a periodic signal.

Periodic convolution

Exercise

Determine the periodic convolution of the signal $f_0(x)$, whose period is 1, defined on a period as $f_0(x) = x$ $(0 < x \leq 1)$ and $g_0(x) = \cos(2\pi x)$. It is strongly advised to firstly sketch the signals.

Periodic convolution

Solution

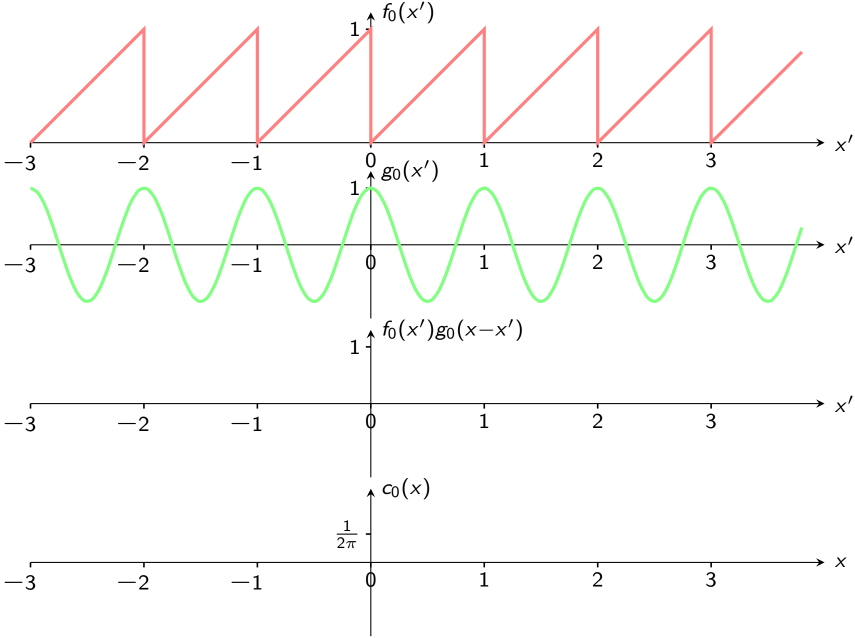
$$c_0(x) = \int_0^1 x' \cos\left(2\pi\left(x - x'\right)\right) dx' = \cos(2\pi x) \int_0^1 x' \cos\left(2\pi x'\right) dx' + \sin(2\pi x) \int_0^1 x' \sin\left(2\pi x'\right) dx'$$

where the trigonometric identity $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ has been applied. Both integrals can be found by integrating by parts:

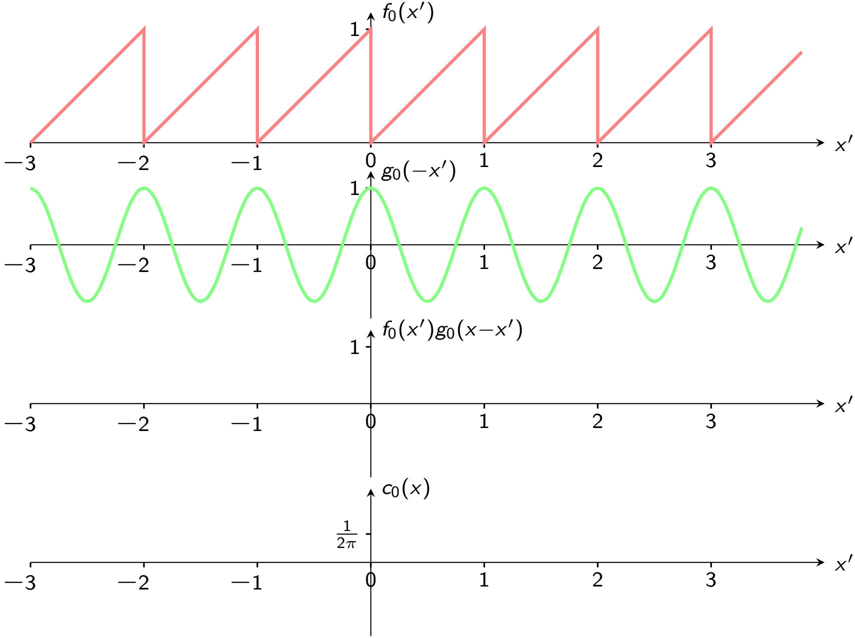
$$\int_0^1 x' \cos\left(2\pi x'\right) dx' = \frac{1}{2\pi} x' \sin(2\pi x')\Big|_0^1 - \frac{1}{2\pi} \int_0^1 \sin\left(2\pi x'\right) dx' = 0$$

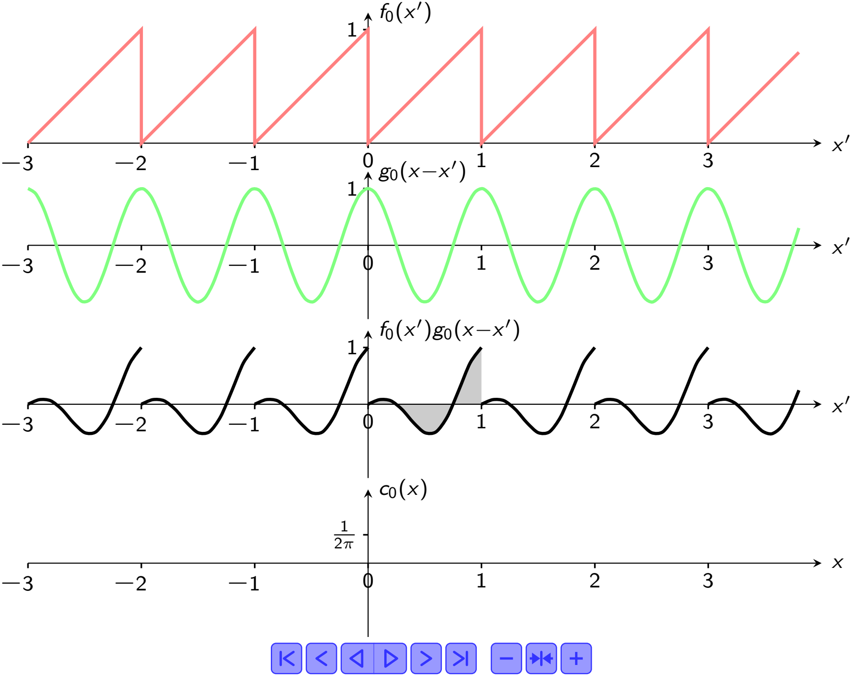
$$\int_0^1 x' \sin\left(2\pi x'\right) dx' = \frac{-1}{2\pi} x' \cos(2\pi x')\Big|_0^1 + \frac{1}{2\pi} \int_0^1 \cos\left(2\pi x'\right) dx' = -\frac{1}{2\pi}$$

therefore $c_0(x) = -\frac{1}{2\pi}\sin(2\pi x)$



$f_0(x')$
1
x'
−3 −2 −1 0 1 2 3
$g_0(x')$
1
x'
−3 −2 −1 0 1 2 3
$f_0(x')g_0(x-x')$
1
x'
−3 −2 −1 0 1 2 3
$c_0(x)$
$\frac{1}{2\pi}$
x
−3 −2 −1 0 1 2 3


$f_0(x')$
1
x'
-3 -2 -1 0 1 2 3
$g_0(-x')$
1
x'
-3 -2 -1 0 1 2 3
$f_0(x')g_0(x-x')$
1
x'
-3 -2 -1 0 1 2 3
$c_0(x)$
$\frac{1}{2\pi}$
x'
-3 -2 -1 0 1 2 3



$f_0(x')$
1
x'
-3 -2 -1 0 1 2 3
$g_0(x-x')$
1
x'
-3 -2 -1 0 1 2 3
$f_0(x')g_0(x-x')$
1
x'
-3 -2 -1 0 1 2 3
$c_0(x)$
$\frac{1}{2\pi}$
x
-3 -2 -1 0 1 2 3

Periodic convolution

Exercise

Determine the periodic convolution of the signals $f_0(x)$, whose period is 1, and it is defined as $f_0(x) = e^x$ $(0 < x \leq 1)$ and $g_0(x) = \cos(2\pi x)$. It is strongly advised to firstly sketch the signals.

Properties of convolution

In the case of well-behaved functions, convolution (in any of the previous forms) satisfies the following properties:

Commutative property $a(\tau) * b(\tau) = b(\tau) * a(\tau)$

Associative property $a(\tau) * [b(\tau) * c(\tau)] = [a(\tau) * b(\tau)] * c(\tau)$

Distributive property $a(\tau) * [b(\tau) + c(\tau)] = a(\tau) * b(\tau) + a(\tau) * c(\tau)$

In engineering, it is usually safe to suppose that the previous properties are satisfied.

Properties of convolution

Exercise

Prove the commutative, associative and distributive properties in case of well-behaved functions (e.g., functions meeting the conditions of the Fubini-Tonelli theorem).

In addition, prove the two following properties:

a) If $c(\tau) = a(\tau) * b(\tau)$, then $c(\tau - \tau_0) = a(\tau - \tau_0) * b(\tau) = a(\tau) * b(\tau - \tau_0)$ (Translational equivariance property)

b) $[\alpha a(\tau)] * b(\tau) = a(\tau) * [\alpha b(\tau)] = \alpha [a(\tau) * b(\tau)]$,

Outline

1. Introduction
2. Mapping between signal spaces
3. Systems
4. The convolution operator
5. The Correlation Operator
   - Crosscorrelation
   - Example

Definition of the crosscorrelation

- In order to measure the similarity between two signals, the crosscorrelation, or just crosscorrelation, is tailored.
- In the case of continuous-variable and nonperiodic signals, this operation between two signals $f(x)$ and $g(x)$ is defined as:

$$\boxed{R_{fg}(x) = \int_{-\infty}^{\infty} f(x+x')g^*(x')dx' = \langle f(x'+x), g(x')\rangle}$$

- This operation is a more sophisticated measure than the ordinary scalar product since it detects shifted similarities or differences.
- It is simple to show that the previous operation can be express in terms of the convolution by means of a change of variable $(y' = x + x')$:

$$\boxed{R_{fg}(x) = \int_{-\infty}^{\infty} f(y')g^*(y'-x)dy' = f(x) * g^*(-x)}$$

Undertanding the crosscorrelation

- From a mathematical point of view, it can be understood as a convolution without the need of reversing the second signal (and considering its complex conjugate).
- In order to understand why the correlation measures the similarity between signals, in addition to the expression in terms of the inner product, it is also interesting to study the difference signal:

$$z(x) = f(x + x') - g(x)$$

- The energy of this difference signal is:

$$E[z(x)] = E[f(x)] + E[g(x)] - 2\Re\{R_{fg}(x)\}$$

- The higher the crosscorrelation, the less the energy of the difference $z(x)$, i.e., more similar are $f(x + x')$ and $g(x)$.
- The less the crosscorrelation, the higher the energy of the difference $z(x)$, i.e., less similar are $f(x + x')$ and $g(x)$.

Definition of the autocorrelation

- The autocorrelation is defined when both signals are the same one:

$$\boxed{R_f(x) = \int_{-\infty}^{\infty} f(x+x')f^*(x')dx' = \langle f(x'+x), f(x') \rangle}$$

- This operation enable us to measure the similarity of a signal with a shifted copy of itself, i.e., it gives us an idea of the changes of $f(x)$.
- For example, if $|R_f(x_0)|$ is large, then $f(x)$ and $f(x+x_0)$ are similar.

Properties of the correlation

- The correlation meets the following properties:

$R_{fg}(x) = R^*_{gf}(-x)$
$R_f(x) = R^*_f(-x)$ Hermitian signal
$z(x) = f(x) \pm g(x) \Rightarrow R_z(x) = R_f(x) + R_g(x) \pm [R_{fg}(x) + R_{gf}(x)]$
$R_f(0) = E[f(x)]$
Schwarz's inequality: $\lvert R_{fg}(x)\rvert^2 \leq E[f(x)]\,E[g(x)] \qquad \lvert R_f(x)\rvert \leq E[f(x)]$

Uncorrelation, orthogonality and incoherence

- The definition o **uncorrelated**, **orthogonal** and **incoherent** signals are similar but there are small details to take into account.

$$z(x) = f(x) + g(x) \Rightarrow R_z(x) = R_f(x) + R_g(x) + [R_{fg}(x) + R_{gf}(x)]$$
$$E[z(x)] = E[f(x)] + E[g(x)] + [R_{fg}(0) + R_{gf}(0)]$$

- **Uncorrelation** between $f(x)$ and $g(x)$

$$R_{fg}(x) = R_{gf}(x) = 0,\ \forall x$$

- **Orthogonality** between $f(x)$ and $g(x)$

$$R_{fg}(0) = R_{gf}(0) =< f(x), g(x) >= 0$$

$$\boxed{E[z(x)] = E[f(x)] + E[g(x)]}$$

- **Incoherence** between $f(x)$ and $g(x)$

$$R_{fg}(0) + R_{gf}(0) = 2\Re\{R_f g(0)\} = 0$$

$$R_{fg}(0) = R^*_{gf}(0) \Rightarrow \Re\{R_{fg}(0)\} = \Re\{R_{gf}(0)\}$$
$$R_{fg}(0) = R^*_{gf}(0) \Rightarrow \Im\{R_{fg}(0)\} = -\Im\{R_{gf}(0)\}$$

Correlation on other signal spaces

- Similar definitions can be analogously given on other signal spaces:

$$R_{fg}(x) = \int_{-\infty}^{\infty} f(x+x')g^*(x')dx' = \left\langle f(x'+x), g(x') \right\rangle$$

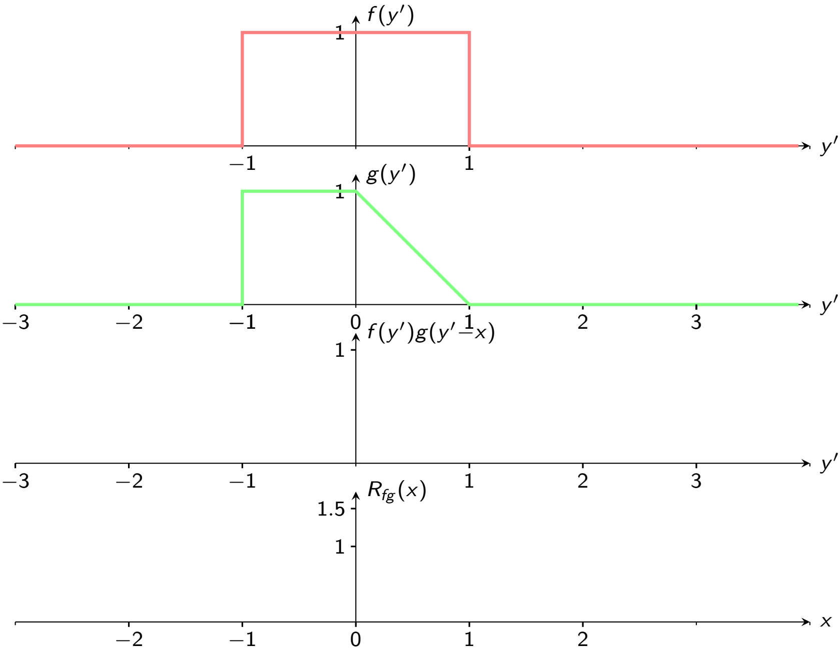
$$R_{f_0 g_0}(x) = \int_{\langle X_0 \rangle} f_0(x+x')g_0^*(x')dx' = \left\langle f_0(x'+x), g_0(x') \right\rangle$$

$$R_{xy}(n) = \sum_{m=-\infty}^{\infty} x(n+m)y^*(m) = \langle x(n+m), y(m) \rangle$$

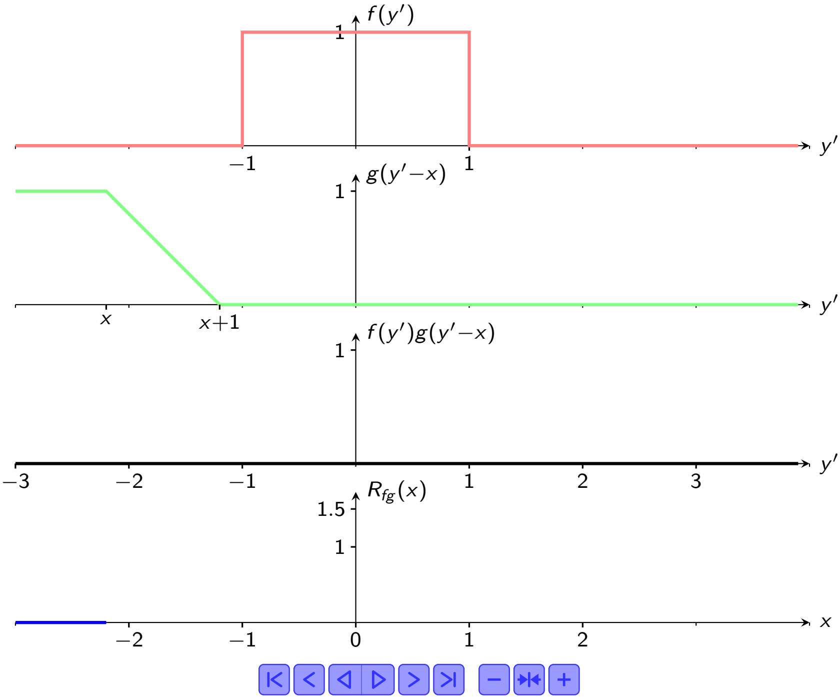
$$R_{x_0 y_0}(n) = \sum_{m=\langle N_0 \rangle} x_0(n+m)y_0^*(m) = \langle x_0(n+m), y_0(m) \rangle$$

- The properties of these correlations are identical to the previously described for $R_{fg}(x)$.

- It is relevant to note that some authors prefer a slightly different definition of the correlation (e. g., $\cancel{R_{fg}(x) = \int_{-\infty}^{\infty} f^*(x')g(x+x')dx'}$). However, the concepts are equivalent.



$f(y')$
$g(y')$
$f(y')g(y'-x)$
$R_{fg}(x)$
y'
x
1
1.5
−3
−2
−1
0
1
2
3



$f(y')$
1
-1
1
y'
$g(y'-x)$
1
x
$x+1$
y'
$f(y')g(y'-x)$
1
-3
-2
-1
1
2
3
y'
$R_{fg}(x)$
1.5
1
-2
-1
0
1
2
x

Exercise

Exercise

Determine the analytical expression for the crosscorrelation of the previous animated example.

Solution

$$R_{fg}(x) = \begin{cases} 0 & x \leq -2 \\ 2 + 2x + \frac{x^2}{2} & -2 \leq x \leq -1 \\ \frac{3}{2} + x & -1 \leq x \leq 0 \\ \frac{3}{2} - \frac{x^2}{2} & 0 \leq x \leq 1 \\ 2 - x & 1 \leq x \leq 2 \\ 0 & 2 \leq x \end{cases}$$

Example

Faces generated by artificial intelligence (convolutional neural networks, CNN)