Signal spaces

Chapter 2

Jaime Laviada

Área de Teoría de la Señal y Comunicaciones
Universidad de Oviedo

Outline I

Outline II

4. Additional signal classifications
   - Right- and left-handed signals
   - Real- and imaginary-valued signals
   - Even and odd signals
   - Hermitian and anti-Hermitian signals

5. Signal energy, power and mean

6. Basic operations with signals

Outline

Introduction

- In order to understand the concept of space signal, the concept of signal must be firstly introduced.

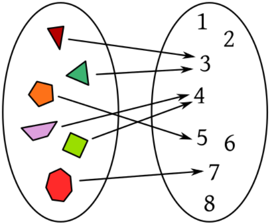
What is a signal?

Any function of one or several variables representing any information regarding the behavior or nature of a physical phenomenon, which can be stored, represented or handled.

- Key features:
  - From a mathematical point of view, they are represented as functions.
  - They are related to some phenomenon.
  - They convey some information.

So... what is a function?

- It is of interest to review the concept of function. What is a function?
- A function is a mapping, which allows associating each element of a set K_1 (domain) a single element of another set K_2 (codomain).



- From a mathematics point of view, it is represented as:

$$\begin{aligned} a: K_1 &\to K_2 \\ \tau &\to a(\tau) \end{aligned}$$

So... what is a function? II

- From a mathematics point of view, it is represented as:

$$\begin{aligned} a:\, K_1 &\to K_2 \\ \tau &\mapsto a(\tau) \end{aligned}$$

- In this course, the sets K_1 and K_2 are sets of numbers.
  - The most common sets will be the natural, integer, real and complex numbers, denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$.
  - Identical concepts can be extended to multivariate functions considering sets such as $\mathbb{N}\times\mathbb{N}\equiv\mathbb{N}^2$ or equivalent ones.
    - In these cases, there are several variables (i.e., arguments) (τ_1, τ_2).

So... what is a function? III

- From a mathematical point of view, it is represented as:

$$\begin{aligned} a: K_1 &\to K_2 \\ \tau &\mapsto a(\tau) \end{aligned}$$

- The variable τ is referred to as independent variable:
  - In those cases where the domain (i.e., K_1) is $\mathbb{Z}$, the independent variable and function will be denoted as $\tau \equiv n$ and $x(n)$, respectively.
  - In those cases where the domain (i.e., K_1) is $\mathbb{R}$, in general, the independent variable and function will be denoted as $\tau \equiv x$ and $f(x)$, respectively.
    - This independent variable usually represents time or space enabling an even more specific notation $x \equiv t$ or $x \equiv z$.

Kind of signals

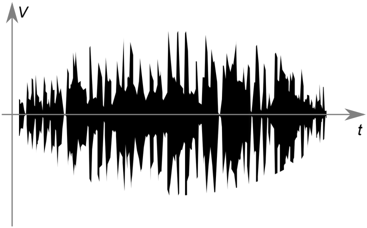
- It is possible to group signals based on multiple considerations:
  - Continuous-variable (CV) signals - discrete-variable signals (DV)
  - Real-valued signals - Complex-valued signals
  - Univariate signals - multivariate signals
  - Scalar signals - Vector-valued signals

Continuous- and discrete-variable signals

- The classification between continuous- and discrete-variable (CV and DV) signals is based on the nature of the independent variable:

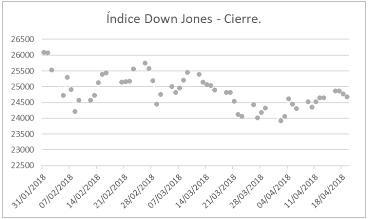
- Continuous-variable: the domain K_1 is not countable.
- In general, $\tau \in \mathbb{R}$, then $\tau \equiv x$.

Example: Voice signal



- Discrete-variable: the domain K_1 is countable.
- In general, $\tau \in \mathbb{Z}$, then $\tau \equiv n$.

Example: Stock market index

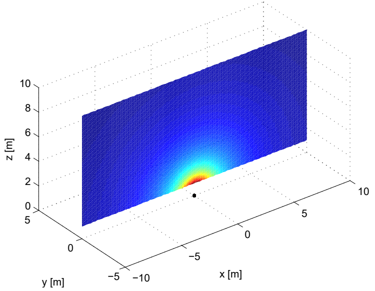


Scalar and vector signals

- The classification between scalar and vector signals depends on the physical phenomenon represented by $a(\tau)$.

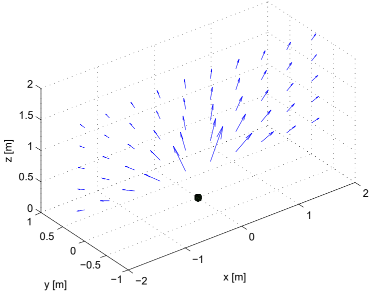
- Scalar signal: the codomain, K_2, contains only scalars (e.g., $\mathbb{R}$ o $\mathbb{C}$)

  Example: Electric potential



- Vector signal: the codomain, K_2, contains only n-tuples (e.g., $\mathbb{R}^3$ o $\mathbb{C}^3$)

  Example: Electric field

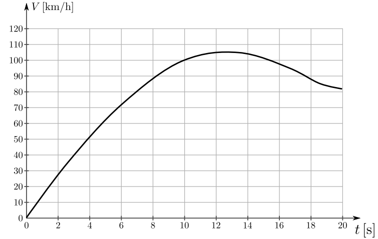


- Only scalar signals will be considered in this course.

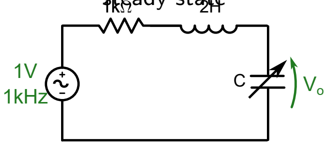
Real- and complex-valued

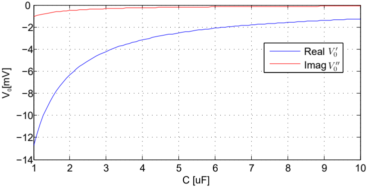
- The classification between real- and complex-valued signals depends on the codomain K_2.

Real-valued: $K_2 \subseteq \mathbb{R}$
Example: Vehicle speed



Complex-valued signal: $K_2 \subseteq \mathbb{C}$
Example: Voltage as a phasor under AC steady state





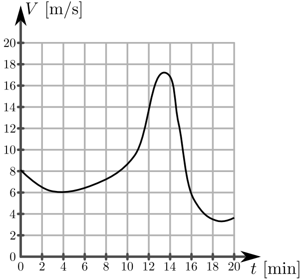
- In the telecommunications field, passband transmissions and electromagnetic fields are usually represented by means of complex-valued signals.

Univariate and multivariate signals

- The classification between univariate and multivariate signals depends on the number of independent variables:

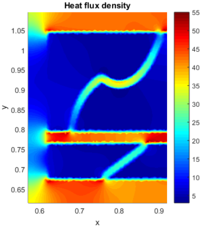
- Univariate signal: the signal depends on a single variable $a(\tau)$

  Example: Wind speed



- Multivariate signal: the signal depends on several independent variables. For example, $a(\tau_1, \tau_2)$.

  Example: Heat flux density



- In physics, a field (i.e., a time-space distribution of a physical quantity) is always a multivariate signal.

Outline

Why do we study signal spaces?

- In order to understand the Theory of Transforms, it is necessary to know the concept of signal space.
- They are a kind of vector space and, therefore, vector space concepts can be applied.
- What is a vector space?
  - It is a collection of objects, called vectors, on which several operations are defined.
  - This collection of vectors will be denoted by $\mathcal{V}$.
  - Some operations also required the use of a field, which will be denoted by $\mathcal{K}$. The elements of this field are called scalars.
  - This vector space is referred to as vector space over $\mathcal{K}$ and it is denoted by $\mathcal{V}_{\mathcal{K}}$.

Vector space

The following two operations are required to qualify $\mathcal{V}$as a vector space:

- Vector addition:

$$\begin{aligned} + : \mathcal{V} \times \mathcal{V} &\to \mathcal{V} \\ (\vec{a}, \vec{b}) &\mapsto \vec{c} = \vec{a} + \vec{b} \end{aligned}$$

this operation must satisfy the commutativity and associativity of addition as well as the existence of the identity and inverse element denoted by $\vec{0}$ and $-\vec{a}$.

- Scalar multiplication:

$$\begin{aligned} \cdot : \mathcal{K} \times \mathcal{V} &\to \mathcal{V} \\ (\alpha, \vec{a}) &\mapsto \vec{b} = \alpha \vec{a} \end{aligned}$$

this operation must satisfy the associativity of scalar multiplication with field multiplication, the existence of the identity element, distributivity of scalar multiplication over field addition and distributivity of scalar multiplication with respect to vector addition.

Vector space

- The previous operations can be performed on functions/signals and, consequently, it enables to define vector space in which the vectors are functions/signals.

$$\boxed{\vec{a} \equiv f(x) \text{ ó } \vec{a} \equiv x(n)}$$

Relevant definitions on vector spaces

The following additional operations can be (optionally) defined on a vector space:

- Norm.
- Distance.
- Inner product (dot product).

In general, it will be supposed that the scalar field is either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$ hereinafter.

Relevant definitions on vector spaces

Norm: it (conceptually) enables to measure the size or length of the vectors

$$\begin{aligned} \|\cdot\| : \mathcal{V}_{\mathcal{K}} &\to \mathbb{R} \\ \vec{a} &\mapsto \alpha = \|\vec{a}\| \end{aligned}$$

This operation must satisfy the following requirements:

- $\|\vec{a}\| \geq 0$ and $\|\vec{a}\| = 0 \Leftrightarrow \vec{a} = \vec{0}$
- $\left\|\vec{a} + \vec{b}\right\| \leq \|\vec{a}\| + \left\|\vec{b}\right\|$ (triangle inequality)
- $\|\alpha\vec{a}\| = |\alpha| \, \|\vec{a}\|$, for any $\alpha \in \mathcal{K}$.

Relevant definitions on vector spaces

Metric: it is a mapping that defines the concept of distance between any two vectors. Consequently, it enables to tell if two elements are equal or not.

$$\begin{aligned} d : \mathcal{V}_{\mathcal{K}} \times \mathcal{V}_{\mathcal{K}} &\to \mathbb{R} \\ (\vec{a}, \vec{b}) &\mapsto \alpha = d(\vec{a}, \vec{b}) \end{aligned}$$

This operation must satisfy the following requirements:

- $d(\vec{a}, \vec{b}) \geq 0$
- $d(\vec{a}, \vec{b}) = d(\vec{b}, \vec{a})$
- $d(\vec{a}, \vec{a}) = 0$

Note that any norm can induce a metric (called metric induced by the norm):

$$d(\vec{a}, \vec{b}) = \left\| \vec{a} - \vec{b} \right\|$$

Inner product - Definition

- Another very relevant operation is the inner product:

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathcal{V}_{\mathcal{K}} \times \mathcal{V}_{\mathcal{K}} &\rightarrow \mathbb{C} \\ \left(\vec{a}, \vec{b}\right) &\mapsto \alpha = \left\langle \vec{a}, \vec{b} \right\rangle \end{aligned}$$

- This operation must satisfy the following requirements:
  - Left-linearity: $\left\langle \alpha\vec{a} + \beta\vec{b}, \vec{c} \right\rangle = \alpha \langle \vec{a}, \vec{c} \rangle + \beta \left\langle \vec{b}, \vec{c} \right\rangle$
  - Conjugate right-linearity: $\left\langle \vec{a}, \beta\vec{b} + \gamma\vec{c} \right\rangle = \beta^* \left\langle \vec{a}, \vec{b} \right\rangle + \gamma^* \langle \vec{a}, \vec{c} \rangle$
  - Conjugate symmetry (Hermitian symmetry): $\left\langle \vec{a}, \vec{b} \right\rangle = \left\langle \vec{b}, \vec{a} \right\rangle^*$
  - Positive-definite: $\langle \vec{a}, \vec{a} \rangle \geq 0$, and $\langle \vec{a}, \vec{a} \rangle = 0 \Leftrightarrow \vec{a} = \vec{0}$
- Inner products are also known as dot products when vectors are *n*-tuples (Euclidean spaces).

Inner product - Details

- The inner product can induce a norm (norm induced by the inner product):

$$\|\vec{a}\| = \langle \vec{a}, \vec{a} \rangle^{\frac{1}{2}}$$

  unless otherwise stated, this norm and the related metric will be used in this course.
- A vector space equipped with an inner product is known as pre-Hiltbert space.
- The scalar $\alpha = \left\langle \vec{a}, \vec{b} \right\rangle$ is known as the projection of $\vec{a}$ on $\vec{b}$.
  - It (conceptually) measures the amount of $\vec{a}$ in $\vec{b}$.
  - If this projection is 0, then the two vectors are said to be orthogonal to each other, $\vec{a} \perp \vec{b}$.

Linear combinations

- After defining the vector addition and scalar multiplication, it is possible to define the concept of linear combination.

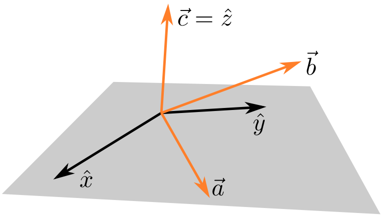
$$\{\vec{a}_m\}_M \Rightarrow \sum_{m=1}^{M} \alpha_m \vec{a}_m = \vec{b} \in \mathcal{V}_{\mathcal{K}}$$

- A collection of M vectors $\{\vec{a}_m\}_M$ is said to be linearly independent if:

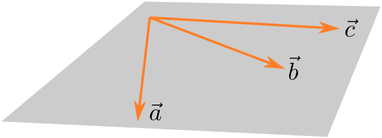
$$\{\vec{a}_m\}_M \text{linearly independent } \Rightarrow \sum_{m=1}^{M} \alpha_m \vec{a}_m = \vec{0} \text{ iff } \alpha_m = 0 \, \forall m$$

in other words, any element of the collection cannot be expressed as a linear combination of the rest of elements.

Linear independence



Linear independence in $\mathbb{R}^3$



Linear dependence in $\mathbb{R}^3$

Bases for vector spaces - I

- A set of M vectors $\{\vec{e}_m\}_M$ from vector space $\mathcal{V}_{\mathcal{K}}$ is called a basis, if:
  - The vectors $\{\vec{e}_m\}_M$ are linearly independent.
  - Every element of $\mathcal{V}_{\mathcal{K}}$ may be written as a linear combination of the elements of the basis

$$\forall \vec{a} \in \mathcal{V}_{\mathcal{K}}, \exists \{\alpha_m\}_M : \ \vec{a} = \sum_{m=1}^{M} \alpha_m \vec{e}_m$$

- The dimension of a vector space is the number of elements of a basis:
  - It can be proved that all the bases have the same number of elements.
  - We will initially suppose that this dimension is finite and, later, this concept will be generalized to infinite-dimensional vector spaces.

Bases for vector spaces - II

- If the elements of a basis $\{\vec{e}_m\}_M$ are mutually orthogonal, then the basis is referred to as orthogonal.
    - If, besides, $\|\vec{e}_m\| = 1$, then the basis is referred to as orthonormal, wherein the norm induced by the inner product is considered.
- When working with orthogonal bases, the coefficients of any linear combination (also known as coordinates) can be calculated as:

$$\vec{a} = \sum_{m=1}^{M} \alpha_m \vec{e}_m, \text{ wherein } \alpha_m = \frac{1}{\|\vec{e}_m\|^2} \langle \vec{a}, \vec{e}_m \rangle$$

- Unless otherwise stated, the norm induced by the inner product will be considered hereinafter.

Infinite-dimensional vector spaces

- If the number of elements of the basis is infinite, the vector space is said to be infinite dimensional
- In these cases, if the basis is countable, it will be denoted by $\{\vec{e}_m\}_M \to \{\vec{e}_m\}$ and the linear combinations are given by:

$$\{\vec{e}_m\} \text{ ortogonal } \to \vec{a} = \sum_{m=-\infty}^{\infty} \alpha_m \vec{e}_m, \text{ with } \alpha_m = \frac{1}{\|\vec{e}_m\|^2} \langle \vec{a}, \vec{e}_m \rangle$$

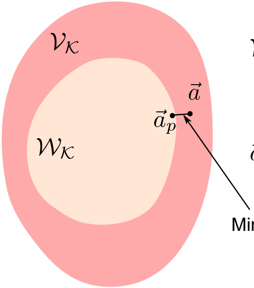
  however, the convergence it is not guaranteed (see next slide).
- If the basis is uncountable, most of the previous concepts can be generalized, however the linear combination cannot be represented by mean of summations:

$$\left\{\vec{e}_\xi\right\} \text{ orthogonal } \to \vec{a} = \mathbf{LC}\left[\alpha_\xi \vec{e}_\xi\right], \text{ with } \alpha_\xi = \frac{1}{\left\|\vec{e}_\xi\right\|^2} \left\langle \vec{a}, \vec{e}_\xi \right\rangle,$$

  wherein **LC** is an operator representing the linear combination of all the elements in the basis and ξ is an index to be defined.

The best approximation theorem

- Let $\mathcal{V}_{\mathcal{K}}$ be a signal space, $\mathcal{W}_{\mathcal{K}}$ a subspace of $\mathcal{V}_{\mathcal{K}}$, and $\{\vec{e}_m\}$ an orthogonal basis for $\mathcal{W}_{\mathcal{K}}$. The best approximation, in terms of distance induced by the norm (also induced by the inner product), for $\vec{a} \in \mathcal{V}_{\mathcal{K}}$ by $\vec{a}_p = \sum_m \alpha_m \vec{e}_m \in \mathcal{W}_{\mathcal{K}}$ is obtained iif:
$$\alpha_m = \frac{1}{\|\vec{e}_m\|^2} \langle \vec{a}, \vec{e}_m \rangle$$
- It is worth noting that the coefficient α_m does not depend on the rest of α_n. Thus, these coefficients are valid after adding or removing new vectors to $\{\vec{e}_m\}$.
- See proof in the appendix.



$$\mathcal{W}_{\mathcal{K}} \subset \mathcal{V}_{\mathcal{K}}$$
$$M = dim(\mathcal{W}_{\mathcal{K}})$$
$$\vec{a}_p = \sum_{m=1}^{M} \alpha_m \vec{e}_m, \text{ with } \alpha_m = \frac{1}{\|\vec{e}_m\|^2} \langle \vec{a}, \vec{e}_m \rangle$$

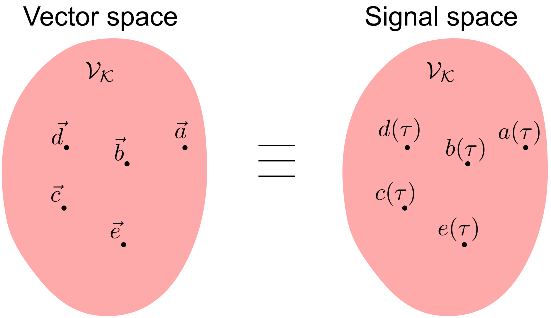
Outline

Signal spaces

- Since the sum of two functions (signals) can be done element by element $b(\tau) = a_1(\tau) + a_2(\tau)$ and the multiplication by a scalar is also defined $b(\tau) = \alpha a(\tau)$, then it is possible to build vector spaces by means of collections of functions

$$\boxed{\vec{a} \equiv a(\tau)}$$

- These vector spaces, called signal spaces, can also be extended by means of inner products, norms and metrics.



Linear combinations for signal spaces

- Three different cases can be considered based on the nature of the signal space.

Finite dimension	Infinite dimension and countable basis	Infinite dimension and uncountable basis
$\{e_m(\tau)\}_M$ orthogonal	$\{e_m(\tau)\}$ orthogonal	$\{e_\xi(\tau)\} = \{e(\tau;\xi)\}$ orthogonal
$a(\tau) = \sum_{m=1}^{M} \alpha_m e_m(\tau)$	$a(\tau) = \sum_{m=-\infty}^{\infty} \alpha_m e_m(\tau)$	$a(\tau) = \int_\xi \alpha(\xi) e(\tau;\xi) d\xi$
$\alpha_m = \frac{1}{\\|e_m(\tau)\\|^2} \langle a(\tau), e_m(\tau)\rangle$	$\alpha_m = \frac{1}{\\|e_m(\tau)\\|^2} \langle a(\tau), e_m(\tau)\rangle$	$\alpha(\xi) = \frac{1}{\\|\vec{e}(\tau;\xi)\\|^2} \langle a(\tau), e(\tau;\xi)\rangle$

- Note that the inner product is still undefined.

Examples of signal spaces - I

Taylor series as basis

It is known that any smooth function $f(x) \in \mathcal{C}^\infty$ can be expanded as:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots = \sum_{m=0}^{\infty} \alpha_m x^m$$

Thus, the monomials x^m are a basis of the signal space of smooth functions since it is straightforward to check that they are linearly independent.

Examples of signal spaces - II

Chebyshev polynomials

The Chebyshev polynomials of first kind are defined as:

$$T_n(x) = \cos\left(n \arccos x\right)$$

It is possible to prove that these polynomials are an orthonormal basis, which allows to expand any continuous and piecewise differentiable, defined on the interval [-1,1]

$$f(x) = \sum_{n=0}^{\infty} \alpha_n T_n(x)$$

- In this course, signals spaces and their corresponding algebra will be defined according to the nature of the independent variable (continuous or discrete) as well as to the periodicity or aperiodicity.

Examples of signal spaces - III

Exercise

Calculate one basis for the discrete-variable signals defined on the interval $-3 \leq n \leq 3$

What is the dimension of the space?

Continuous-variable non-periodic signals

- In this course, we will focus on four signal spaces, each of them with their own algebra.
- The first one is related to (complex) signals, which are of continuous-variable and non-periodic defined on the interval $(-\infty, \infty)$
- This space will be denoted by $S(-\infty, \infty)$ or $S(\mathbb{R})$ and, therefore, the independent variable is real.
- Inner product:

$$\boxed{\langle f(x), g(x)\rangle = \int_{-\infty}^{\infty} f(x) g^*(x) dx}$$

Continuous-variable periodic signals

- The second space of interest is composed of continuous-variable periodic signals.

Definition of a CV periodic signal

A signal is periodic if its values are repeated in regular intervals. The shortest of these repetition intervals is called **fundamental period** and, in CV, it will be denoted by X_0.

$$f_0(x + X_0) = f_0(x), \forall x \text{ wherein } X_0 \in \mathbb{R}^+$$

Note that a signal, whose period is X_0, is also a signal with period mX_0, with $\forall m \in \mathbb{N} \setminus \{0\}$.

Continuous-variable periodic signals II

Exercise

Check the validity of vector addition and scalar multiplication for the signal space of CV periodic signals:

1. $g_0(x) = f_{0,1}(x) + f_{0,2}(x),$ wherein $f_{0,1}(x), f_{0,2}(x)$ have a period X_0
2. $g_0(x) = \alpha f_0(x),$ wherein $f_0(x)$ has a period X_0 and $\alpha \in \mathbb{C}$

This signal space will be denoted by $P(X_0)$.

Continuous-variable periodic signals III

Exercise

Suppose one signal $f_{0,1}(x)$ with period $X_{0,1}$ and another one $f_{0,2}(x)$ with period $X_{0,2}$:

1. What are the conditions so $g_0(x) = f_{0,1}(x) + f_{0,2}(x)$ is periodic?
2. If the fundamental periods of the previous signals are $X_{0,1} = mX_0$ and $X_{0,2} = nX_0$, with $X_0 \in \mathbb{R}^+$ and $m, n \in \mathbb{N}^+$ What is the fundamental period of g_0?

Continuous-variable periodic signals III

Exercise

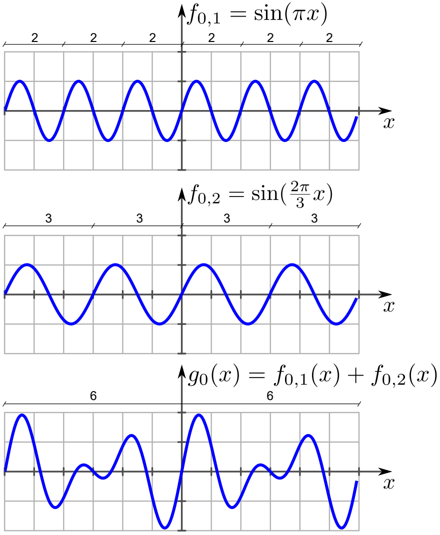
Suppose one signal $f_{0,1}(x)$ with period $X_{0,1}$ and another one $f_{0,2}(x)$ with period $X_{0,2}$:

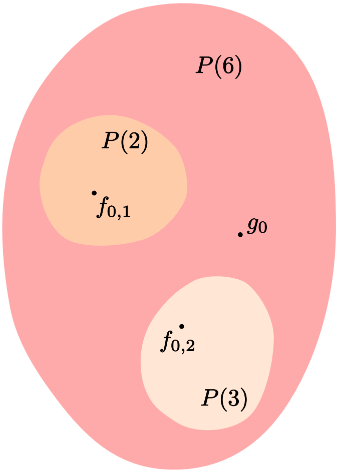
1. What are the conditions so $g_0(x) = f_{0,1}(x) + f_{0,2}(x)$ is periodic?
2. If the fundamental periods of the previous signals are $X_{0,1} = mX_0$ and $X_{0,2} = nX_0$, with $X_0 \in \mathbb{R}^+$ and $m, n \in \mathbb{N}^+$ What is the fundamental period of g_0?

Solution

The signal $f_{0,1}(x)$ has a period $X_{0,1}$ but also $2X_{0,1}$, $3X_{0,1}$, etc. In a similar fashion, the signal $f_{0,2}(x)$ has a period $X_{0,2}$, $2X_{0,2}$, $3X_{0,2}$, etc.
therefore, there must be $p, q \in \mathbb{N}$ so $pX_{0,1} = qX_{0,2}$.
The previous condition is equivalent to requiring that **the ratio between both periods must be a rational number** $\frac{X_{0,1}}{X_{0,2}} \in \mathbb{Q}$
In the second part, the fundamental period of the sum will be $\text{l.c.m}(m, n)X_0$

Continuous-variable periodic signals IV





Continuous-variable periodic signals V

- Hence, the second signal space of interest is composed of the continuous-variable periodic signals, with period X_0.
- This space is denoted by $P(X_0)$ and their signals are defined on the real axis $(-\infty, \infty)$.
  - Inner product:

$$\boxed{\langle f_0(x), g_0(x)\rangle = \int_{\langle X_0\rangle} f_0(x) g_0^*(x) dx}$$

  - Note that $P(X_0) \subseteq P(mX_0)$, $\forall m \in \mathbb{N} \setminus \{0\}$.

Discrete-variable non-periodic signals

- The discrete-variable non-periodic signals are defined in a similar way to the continuous-variable non-periodic signals.
- They are denoted by $D(\mathbb{Z})$ or $D(-\infty, \infty)$ and they are defined on the real axis $(-\infty, \infty)$.
  - Inner product:

$$\boxed{\langle x(n), y(n)\rangle = \sum_{n=-\infty}^{\infty} x(n) y^*(n)}$$

Discrete-variable periodic signals - I

- They are defined in a similar way to the DV periodic signals.
    - Their in-depth study is postponed to later chapters.

Definition of DV periodic signal

A DV signal is periodic if its values are repeated at regular intervals.
The shortest of these intervals is known as **fundamental period** and, in DV, it will be denoted by N_0.

$$x_0(n+N_0) = x_0(n), \forall n \text{ with } N_0 \in \mathbb{N} \setminus \{0\}$$

Note that a signal with period N_0 has also period mX_0, $\forall m \in \mathbb{N} \setminus \{0\}$.

Discrete-variable periodic signals - II

- The signal space for DV periodic signals will be denoted by $D(N_0)$.
- They are defined in a similar fashion to the CV ones.
- In this case, the inner product is defined as:

$$\boxed{\langle x_0(n), y_0(n)\rangle = \sum_{n=\langle N_0\rangle} x_0(n) y_0^*(n)}$$

- Note that, again, $D(N_0) \subseteq D(mN_0)$, with $\forall m \in \mathbb{N} \setminus \{0\}$.

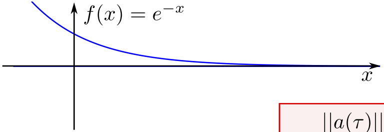
Summary of signal spaces

Continuous-variable

Aperiodic signals

$S(\mathbb{R})$ $\quad \langle f(x), g(x)\rangle = \int_{-\infty}^{\infty} f(x) g^*(x) dx$

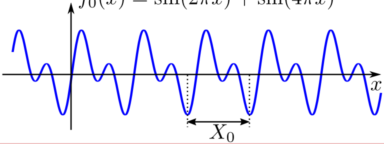
$f(x) = e^{-x}$



Periodic signals

$P(X_0)$ $\quad \langle f_0(x), g_0(x)\rangle = \int_{\langle X_0\rangle} f_0(x) g_0^*(x) dx$

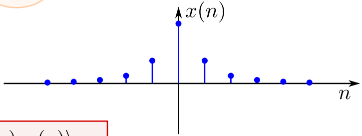
$f_0(x) = \sin(2\pi x) + \sin(4\pi x)$



Discrete-variable

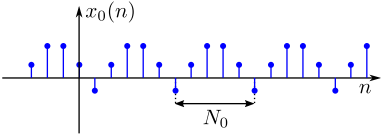
Aperiodic signals

$D(\mathbb{Z})$ $\quad \langle x(n), y(n)\rangle = \sum_{n=-\infty}^{\infty} x(n) y^*(n)$



Periodic signals

$D(N_0)$ $\quad \langle x_0(n), y_0(n)\rangle = \sum_{n=\langle N_0\rangle} x_0(n) y_0^*(n)$



$$||a(\tau)||^2 = \langle a(\tau), a(\tau)\rangle$$

$$d\left(a(\tau), b(\tau)\right) = ||a(\tau) - b(\tau)||$$

Distance between signals - example

Exercise

Sketch the following signals and calculate the distance between them:

$$f(x) = \sin x \quad g(x) = \begin{cases} 2 & x = 0 \\ \sin x & \text{otherwise} \end{cases}$$

Solution

The distance between two signals which are identical "**almost everywhere**" (zero measure, differences at isolated points) is zero. Nevertheless, they are not the same signal despite the distance between them is zero.

Outline

Additional signal classifications

- In addition to the previous classifications, there are many other relevant classifications.
  - Right- and left-sided signals, finite-length and infinite-length signals
  - Real- and imaginary-valued signals.
  - Even and odd signals.
  - Hermitian and anti-Hermitian signals.

Signal classification according to its domain

- Depending on the definition domain, the following classification can be established:

  Finite-length signals: they are defined on a given continuous interval $[a, b]$ or discrete interval $[N_1, N_2]$.

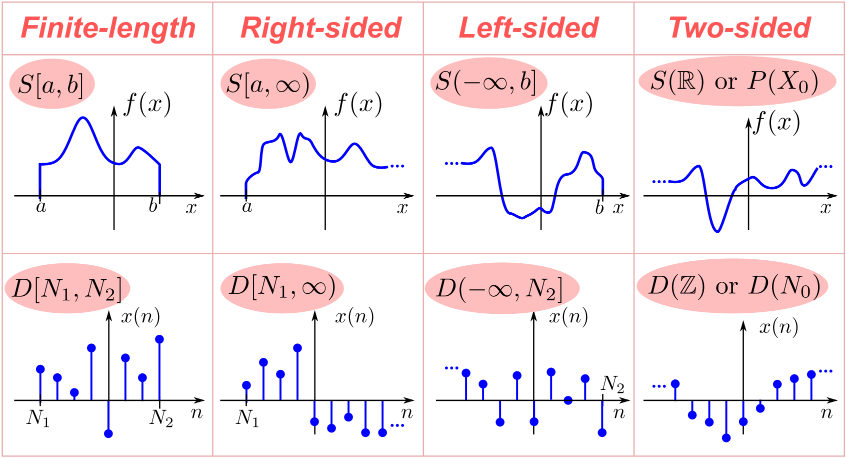
  Right-sided signals: they are defined on a continuous interval $[a, \infty)$ or discrete interval $[N_1, \infty)$.

  Left-sided signals: they are defined on a continuous interval $(\infty, b]$ or discrete interval $(\infty, N_2]$.

  Two-sided signals: they are defined on the entire axis $\mathbb{R}$ or $\mathbb{Z}$.

- Note that the last three cases entail signals with an infinite-length support.
- The previous definitions are also applied using the same criteria but using the support of the function instead of the definition domain.

Signal classification according to its domain



Real- and imaginary-valued signals

Real-valued signal: a real-valued signal is a signal $a_r(\tau)$ such that $a_r(\tau) = a_r^*(\tau)$. In these signals, the imaginary part is zero.

Imaginary-valued signal: a imaginary-valued signal is a signal $a_i(\tau)$ such that $a_i(\tau) = -a^*(\tau)$. In these signals, the real part is zero.

Every signal $a(\tau)$ can be decomposed into the sum of a real- and imaginary-valued signals:

$$a(\tau) = a_r(\tau) + a_i(\tau) \rightarrow \left\{ \begin{array}{c} a_r(\tau) = \frac{1}{2}\left[a(\tau) + a^*(\tau)\right] \\ a_i(\tau) = \frac{1}{2}\left[a(\tau) - a^*(\tau)\right] \end{array} \right\}$$

(see proof in Appendix E.1 of book by Emilio Gago)

Even and odd signals

Even signal: an even signal is a signal $a_e(\tau)$ such that $a_e(\tau) = a_e(-\tau)$.

- These signals have symmetry with respect to the y-axis.

Odd signal: an odd signal is a signal $a_o(\tau)$ such that $a_o(\tau) = -a_o(-\tau)$.

- These signals have rotational symmetry with respect to the origin.

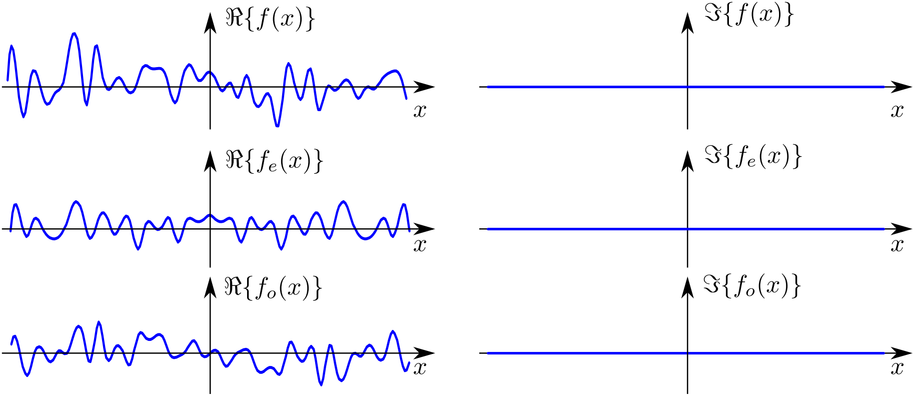
Every signal $a(\tau)$ can be decomposed into an odd and even part:

$$a(\tau) = a_e(\tau) + a_o(\tau) \rightarrow \left\{ \begin{array}{l} a_e(\tau) = \frac{1}{2}\left[a(\tau) + a(-\tau)\right] \\ a_o(\tau) = \frac{1}{2}\left[a(\tau) - a(-\tau)\right] \end{array} \right\}$$

(see proof in Appendix E.1 of book by Emilio Gago)

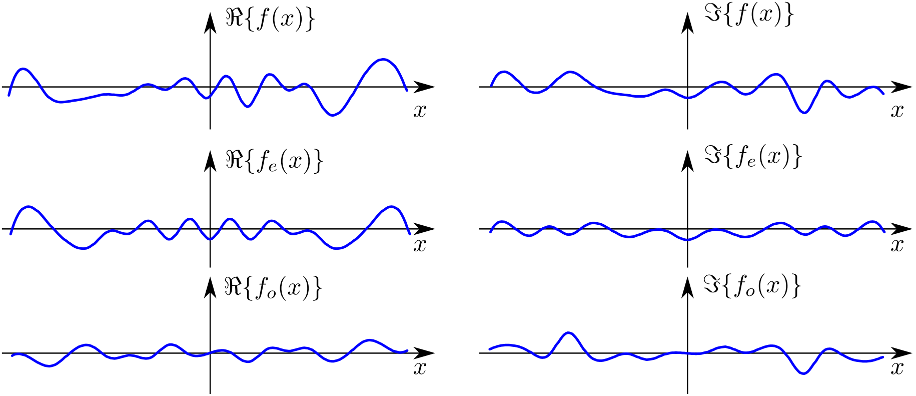
Even and odd signals

Example of even/odd decomposition for a real-valued signal.



Even and odd signals

Example of even/odd decomposition for a complex-valued signal.



Even and odd signals

Exercise

Show that every continuous-variable, differentiable and even signal satisfies $f'(0) = 0$.

Exercise

Show that every continuous-variable and odd signal satisfies $f(0) = 0$.

Even and odd signals

Exercise

Determine and sketch the real and odd parts of the following discrete-variable signal:

$$x(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Hermitian and anti-Hermitian signals

Hermitian signal: a Hermitian signal is a signal $a_h(\tau)$ such that $a_h(\tau) = a_h^*(-\tau)$.

- Consequently, its real part is even and its imaginary part is odd.

Anti-Hermitian signal: an anti-Hermitian signal is a signal $a_{ah}(\tau)$ such that $a_{ah}(\tau) = -a_{ah}^*(-\tau)$.

- Consequently, its real part is odd and its imaginary part is even.

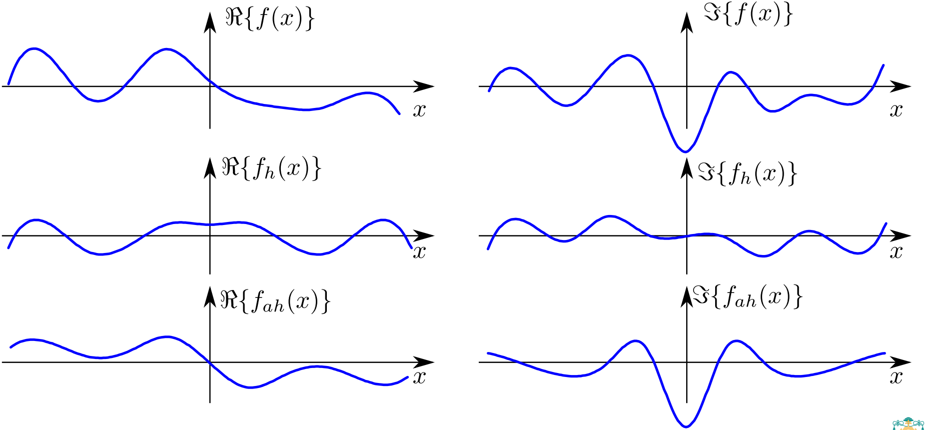
Every signal $a(\tau)$ can be decomposed into the sum of a Hermitian and anti-Hermitian signal:

$$a(\tau) = a_h(\tau) + a_{ah}(\tau) \rightarrow \left\{ \begin{array}{c} a_h(\tau) = \frac{1}{2}\left[a(\tau) + a^*(-\tau)\right] \\ a_{ah}(\tau) = \frac{1}{2}\left[a(\tau) - a^*(-\tau)\right] \end{array} \right\}$$

(see proof in Appendix E.1 of book by Emilio Gago)

Hermitian and anti-Hermitian signals

Example of decomposition of a complex-valued signal into Hermitian and anti-Hermitian parts.



Hermitian and anti-Hermitian signals

Exercise

Show that if $a_h(\tau)$ is **Hermitian**, then **its modulus** $|a_h(\tau)|$ **is an even signal and its phase** $\varphi_{a_h}(\tau)$ **is an even signal**.

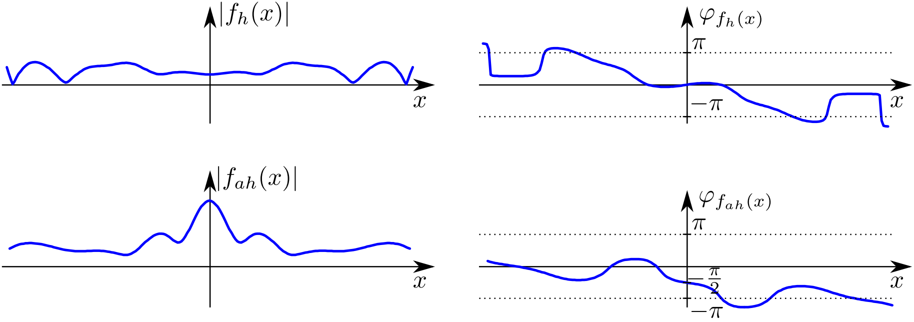
Exercise

Show that if the **anti-Hermitian** signal $a_{ah}(\tau)$ is considered, then **its modulus** $|a_{ah}(\tau)|$ **is an even signal and its phase** $\varphi_{a_{ah}}(\tau)$ **satisfy** the next property:

$$\varphi_{a_{ah}}(\tau) = \pm\pi - \varphi_{a_{ah}}(-\tau).$$

Hermitian and anti-Hermitian signals

Modulus and (unwrapped) phase of the Hermitian and anti-Hermitian signals from the previous example.



Hermitian and anti-Hermitian signals

Exercise

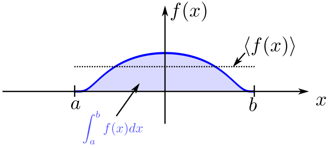
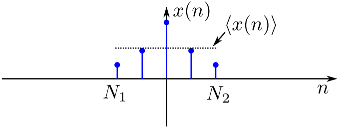
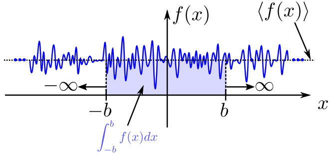
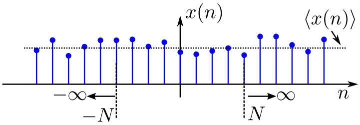
Show that every anti-Hermitian signal has only two possible values for its phase at $x = 0$, i.e., $\varphi_{ah}(0)$. What are those values?

Outline

Signal energy, power and mean

- In addition to the period of a signal, there are several parameters of interest such as:
  - Mean, $\langle a(\tau)\rangle$
  - Instantaneous power, $p_a(\tau) = \mathbf{P}[a(\tau)]$
  - Energy, $E\,[a(\tau)]$
  - Average power, $\langle \mathbf{P}[a(\tau)]\rangle$

Signal average

	Continuous-variable	**Discrete-variable**
Finite-length	$S(a,b)$ $\langle f(x)\rangle = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$ 	$D[N_1,N_2]$ $\langle x(n)\rangle = \dfrac{1}{N_2-N_1+1}\displaystyle\sum_{n=N_1}^{N_2} x(n)$ 
Infinite-length	$S(\mathbb{R})$ $\langle f(x)\rangle = \displaystyle\lim_{b\to\infty}\dfrac{1}{2b}\int_{-b}^{b} f(x)dx$ 	$D(\mathbb{Z})$ $\langle x(n)\rangle = \displaystyle\lim_{N\to\infty}\dfrac{1}{2N+1}\sum_{n=-N}^{N} x(n)$ 

Signal average

- In the case of periodic signals and, therefore, infinite-length signals, the previous definition remains valid but it can be simplified since it is easy to prove that:

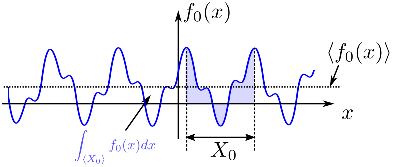
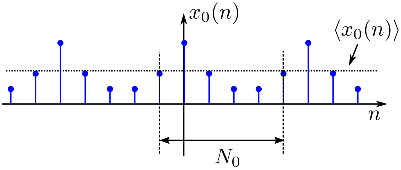
$$\boxed{\langle f_0(x)\rangle = \lim_{b\to\infty}\frac{1}{2b}\int_{-b}^{b} f_0(x)dx = \frac{1}{X_0}\int_{\langle X_0\rangle} f_0(x)dx}$$

$$\boxed{\langle x_0(n)\rangle = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} x(n) = \frac{1}{N_0}\sum_{\langle N_0\rangle} x_0(n)}$$

- In the case of complex-valued signals, it is also trivial to show that:

$$\langle a(\tau)\rangle = \Re\{\langle a(\tau)\rangle\} + j\Im\{\langle a(\tau)\rangle\} \in \mathbb{C}$$

Signal average

	Continuous-variable	Discrete-variable
Infinite-length	$P(X_0)$ $\langle f_0(x)\rangle = \frac{1}{X_0}\int_{\langle X_0\rangle} f_0(x)dx$ 	$D(N_0)$ $\langle x_0(n)\rangle = \frac{1}{N_0}\sum_{\langle N_0\rangle} x_0(n)$ 

Signal average

Exercise

Determine the mean of the following complex-valued signal:

$$f(x) = |\sin(2\pi x)| + j\,|\cos(4\pi x)|$$

Power and energy - CV

- Response of most of real-world systems depends on the signal energy rather than on instantaneous values.
- Power is the amount of energy transferred or converted per unit time.
- In the case of continuous-variable signals:

Quantity	Finite-length	Infinite-length
Instantaneous power	$\mathbf{P}\left[f(x)\right] = \lvert f(x)\rvert^2 = f(x)f^*(x)$	
Energy	$E\left[f(x)\right] = \int_a^b \mathbf{P}\left[f(x)\right] dx$	$E\left[f(x)\right] = \int_{-\infty}^{\infty} \mathbf{P}\left[f(x)\right] dx$
Average power	$\langle \mathbf{P}\left[f(x)\right]\rangle = \frac{1}{b-a}\int_a^b \mathbf{P}\left[f(x)\right] dx$	$\langle \mathbf{P}\left[f(x)\right]\rangle = \lim_{b\to\infty} \frac{1}{2b}\int_b^b \mathbf{P}\left[f(x)\right] dx$

Power and energy - DV

- In the case of discrete-variable signals, the formulation is analogous:

Quantity	Finite-length	Infinite-length
Instantaneous power	$\mathbf{P}\left[x(n)\right] = \lvert x(n)\rvert^2 = x(n)x^*(n)$	
Energy	$E\left[x(n)\right] = \sum_{n=N_1}^{N_2} \mathbf{P}\left[x(n)\right]$	$E\left[x(n)\right] = \sum_{n=-\infty}^{\infty} \mathbf{P}\left[x(n)\right]$
Average power	$\langle \mathbf{P}\left[x(n)\right]\rangle = \frac{1}{N}\sum_{n=N_1}^{N_2} \mathbf{P}\left[x(n)\right]$ $N = N_2 - N_1 + 1$	$\langle \mathbf{P}\left[x(n)\right]\rangle = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=N}^{N} \mathbf{P}\left[x(n)\right]$

Power and energy - periodic signals

- In the case of periodic signals, the average power can be also calculated by operating on a single period:

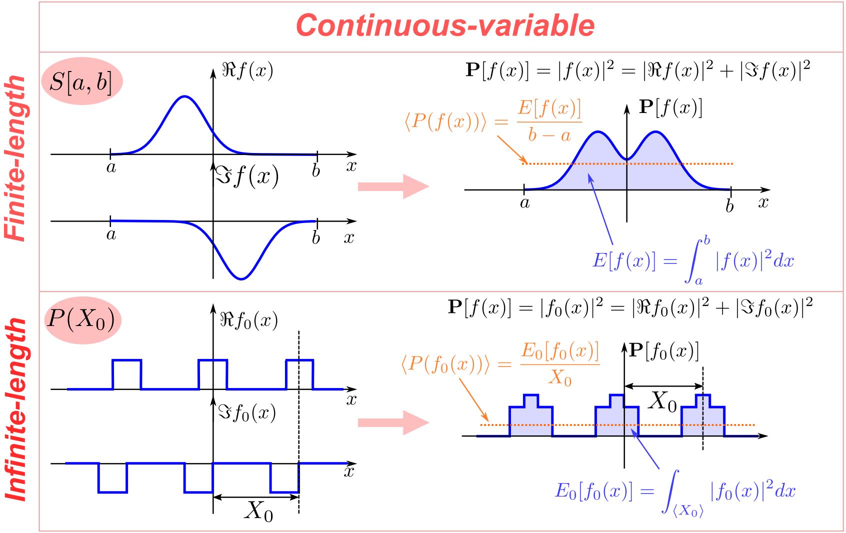
$$\boxed{\langle \mathbf{P}\left[f_0(x)\right]\rangle = \frac{1}{X_0}\int_{\langle X_0\rangle} |f(x)|^2\,dx} \quad \boxed{\langle \mathbf{P}\left[x(n)\right]\rangle = \frac{1}{N_0}\sum_{\langle N_0\rangle} |x(n)|^2}$$

- According to the previous definition, a periodic signal has an infinite energy.
  - However, it is possible to define a quantity called energy per period:

$$E_0[f_0(x)] = \int_{\langle X_0\rangle} |f(x)|^2\,dx \quad E_0[x_0(n)] = \sum_{\langle N_0\rangle} |x(n)|^2 .$$

- **Note**: this definition differs from the definition in the book by Emilio Gago.

Power and energy - periodic signals



Outline

Basic operations with signals

- As it is usual, signal (function) addition and product can be done by pointwise operations:

$$s(\tau) = a(\tau) + b(\tau) \quad p(\tau) = a(\tau)b(\tau)$$

- Moreover, scalar multiplication can also been seen as a pointwise operation with a constant function:

$$b(\tau) = Aa(\tau) \text{ where } A = |A|e^{j\varphi_A}$$

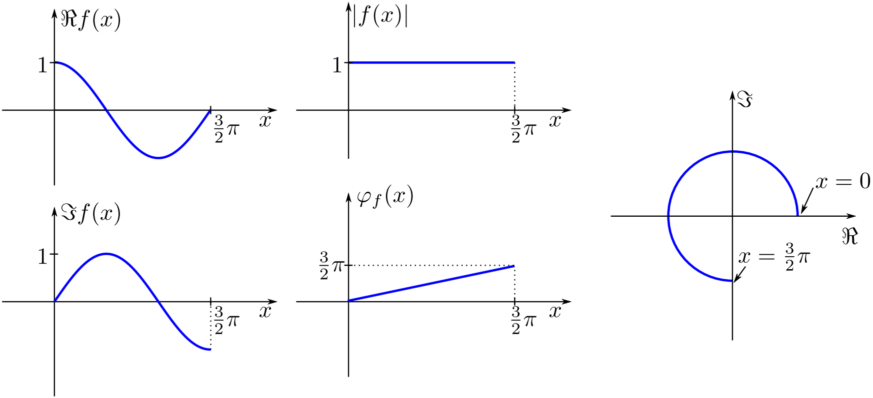
- This operation is usually performed better in terms of modulus and phase:

$$|b(\tau)| = |A||a(\tau)| \quad \varphi_b(\tau) = \varphi_a(\tau) + \varphi_A$$

hence, the signal is amplified ($|A| > 1$) or attenuated ($|A| < 1$) as well as rotated in the complex plane.

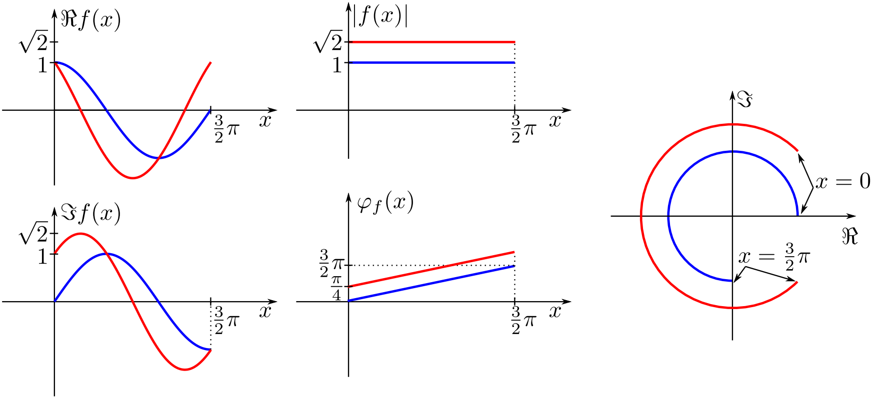
Scalar multiplication

$$f(x) = e^{jx} \text{ con } 0 < x < \tfrac{3}{2}\pi$$



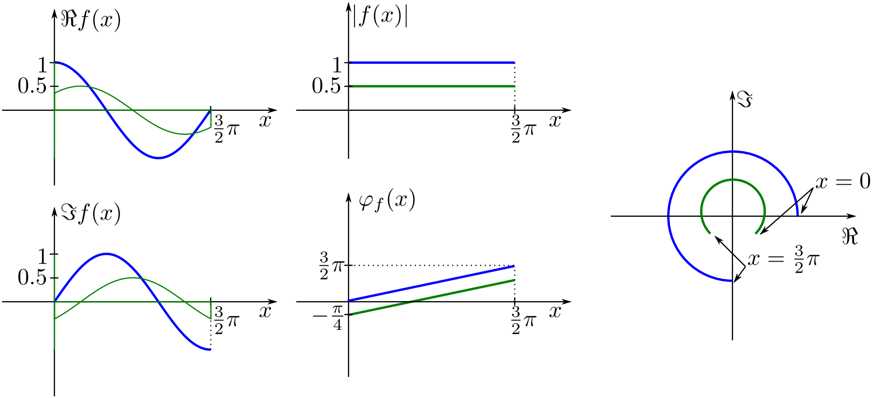
Scalar multiplication

$$g(x) = Af(x) \text{ con } 0 < x < \tfrac{3}{2}\pi,\ A = 1 + j = \sqrt{2}e^{j\frac{\pi}{4}}$$



Scalar multiplication

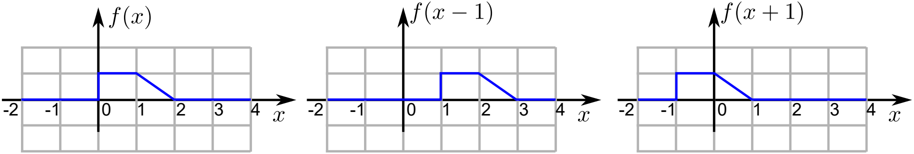
$$g(x) = Af(x) \text{ con } 0 < x < \tfrac{3}{2}\pi,\ A = \tfrac{\sqrt{2}}{4}(1-j) = \tfrac{1}{2}e^{-j\frac{\pi}{4}}$$



Shift

- Shift:

$$a(\tau) \rightarrow b(\tau) = a(\tau - \tau_0)$$



Right shift (delayed signal)

Left shift (advanced signal)

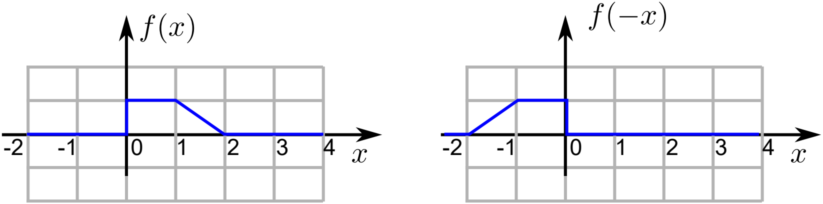
- If the independent variable is time $x \equiv t$, then $t_0 > 0$ represents a delayed signal and $t_0 < 0$ represents an advanced signal.

$\boxed{t_0 > 0 \Rightarrow \text{delayed signal}}$ $\boxed{t_0 < 0 \Rightarrow \text{advanced signal}}$

Reversal

- Reversals are reflections with respect to the axis $\tau = 0$:

$$a(\tau) \rightarrow b(\tau) = a(-\tau)$$

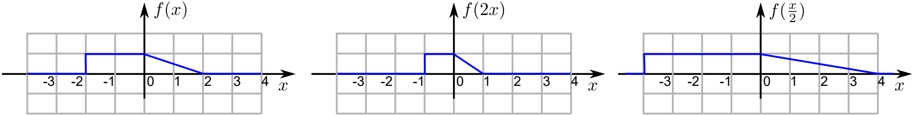


- This operation is defined analogously in DV.

Scaling of the independent variable

- Scaling of the independent variable
  - The DV case will be postponed until Chapter 8.
- In the case of positive scaling:

$$f(x) \to g(x) = f(ax),\ a > 0 \Rightarrow \begin{cases} a > 1 & \text{Compressed signal} \\ a < 1 & \text{Stretched signal} \end{cases}$$



- It is also possible to consider $a < 0$. In this case, the operation is equivalent but followed by a reflection.

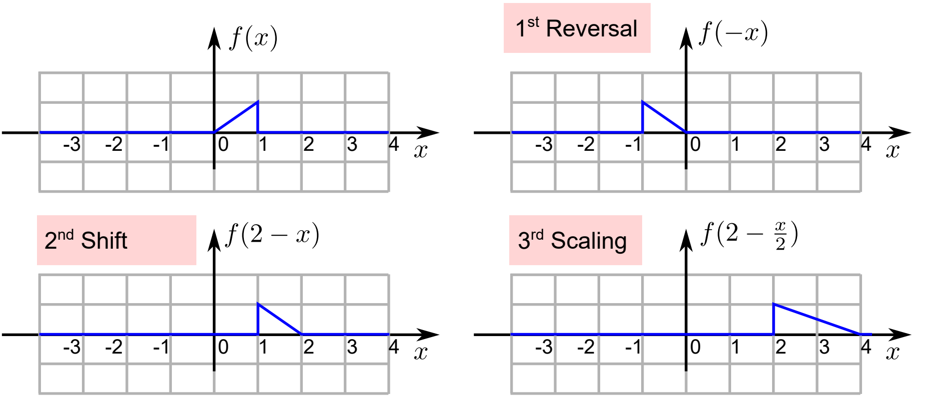
Concatenating operations

- The previous operations can be concatenating yielding a transformation such as:

$$b(\tau) = a(\alpha\tau - \tau_0)$$

- The following order is suggested:
  1. Reversal
  2. Shift
  3. Scaling
- The DV case will be studied in later chapters.

Concatenating operations



Concatenating operations

- It is worthwhile to remember that the following studies have been postponed:
    - Discrete-variable periodic signals.
    - Basic operations with discrete-variable operations: scaling of the independent variable n.

Outline

Appendixes

Proof of the best approximation theorem - I

Let the consider approximation be $\hat{a}(\tau) = \sum_m \alpha_m e_m(\tau)$, where $\{e_m(\tau)\}$ is an orthogonal basis. Thus, the (square) distance is given by:

$$E^2 = d^2\left(a(\tau), \hat{a}(\tau)\right) = \left\|a(\tau) - \hat{a}(\tau)\right\|^2 = \langle a(\tau) - \hat{a}(\tau), a(\tau) - \hat{a}(\tau)\rangle =$$

$$\left\|a(\tau)\right\|^2 + \left\|a(\tau)\right\|^2 - \langle a(\tau), \hat{a}(\tau)\rangle - \langle \hat{a}(\tau), a(\tau)\rangle =$$

$$\left\|a(\tau)\right\|^2 + \sum_m \left|\alpha_m\right|^2 \left\|e_m(\tau)\right\|^2 - \sum_m \alpha_m^* \langle a(\tau), e_m(\tau)\rangle - \sum_m \alpha_m \langle e_m(\tau), a(\tau)\rangle$$

In order to find the optimum value of $\alpha_m = \alpha_{rm} + j\alpha_{im}$, the two following equations must be simultaneously solved:

$$\frac{\partial E^2}{\partial \alpha_{rm}} = 0 \qquad \frac{\partial E^2}{\partial \alpha_{im}} = 0$$

(cont.)

Proof of the best approximation theorem - II

Taking into account that $|\alpha_m|^2 = \alpha_{rm}^2 + \alpha_{im}^2$, then

$$\frac{\partial E^2}{\partial \alpha_{rm}} = 2\alpha_{rm} \|e_m(\tau)\|^2 - \langle a(\tau), e_m(\tau)\rangle - \langle e_m(\tau), a(\tau)\rangle =$$
$$= 2\alpha_{rm} \|e_m(\tau)\|^2 - 2\Re \langle a(\tau), e_m(\tau)\rangle = 0$$
$$\frac{\partial E^2}{\partial \alpha_{im}} = 2\alpha_{im} \|e_m(\tau)\|^2 + j \langle a(\tau), e_m(\tau)\rangle - j \langle e_m(\tau), a(\tau)\rangle =$$
$$= 2\alpha_{rm} \|e_m(\tau)\|^2 - 2\Im \langle a(\tau), e_m(\tau)\rangle = 0$$

and, consequently,

$$\alpha_{rm} = \frac{1}{\|e_m(\tau)\|^2} \Re \langle a(\tau), e_m(\tau)\rangle \qquad \alpha_{im} = \frac{1}{\|e_m(\tau)\|^2} \Im \langle a(\tau), e_m(\tau)\rangle$$

thus,

$$\boxed{\alpha_m = \frac{1}{\|e_m(\tau)\|^2} \langle a(\tau), e_m(\tau)\rangle} \text{(Q.E.D.)}$$

Signal spaces for aperiodic signals in CV

- In this course, several additional space signals will play a key role in some transformations and operations.
- The most generic space will be $S(\mathbb{R})$, which includes the continuous-variable signals, whose support is $\mathbb{R}$.
- $L^p(\mathbb{R})$ is the signal space of signals with support on $\mathbb{R}$, whose p-norm is finite:

$$\|f(x)\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p \, dx\right)^{\frac{1}{p}}$$

$$f(x) \in L^p(\mathbb{R}) \Longleftrightarrow \|f(x)\|_p < +\infty$$

- A relevant case is the ∞-norm:

$$\|f(x)\|_\infty = \lim_{p\to\infty} \|f(x)\|_\infty = \inf\{C \geq 0 : |f(x)| \leq C, \text{ almost everywhere}\}$$

Signal spaces for aperiodic signals in CV

- Two specially relevant spaces are the space absolutely integrable signals $L^1(\mathbb{R})$ and the space of square-integrable signals $L^2(\mathbb{R})$

$$f(x) \in L^1(\mathbb{R}) \Longleftrightarrow \int_{-\infty}^{\infty} |f(x)|\, dx < +\infty$$

$$f(x) \in L^2(\mathbb{R}) \Longleftrightarrow \int_{-\infty}^{\infty} |f(x)|^2\, dx < +\infty$$

- It is worthwhile to observe that $L^1(\mathbb{R})$ is **not** a subspace of $L^2(\mathbb{R})$ and, moreover, $L^2(\mathbb{R})$ is **not** a subspace of $L^1(\mathbb{R})$ either.

Signal spaces for periodic signals in CV

- In the case of CV periodic signals, similar spaces can be also defined by restricting the integral to a single period X_0 yielding $L_p(X_0)$.
- In this case, it is also possible to define signals space for absolutely integrable signals whose period is X_0:

$$L_p^1(X_0) = \left\{ f_0 : \mathbb{R} \to \mathbb{C} \text{such that } f_0 \text{ has a period } X_0 \text{ and } \int_{\langle X_0 \rangle} |f(x)|\, dx < +\infty \right\}$$

- An analogous definition also applies to square-integrable signals with a period X_0:

$$L_p^2(X_0) = \left\{ f_0 : \mathbb{R} \to \mathbb{C} \text{such that } f_0 \text{ has a period } X_0 \text{ and } \int_{\langle X_0 \rangle} |f(x)|^2\, dx < +\infty \right\}$$

- In this case, since the integration interval is bounded, the following statement applies $L_p^2(X_0) \subset L_p^1(X_0)$.

Signal spaced in CV of interest for distributions

- In the study of distributions, some subspaces of $L^1(\mathbb{R})$ will be of interest.
- The Schwarz space $\mathscr{L}(\mathbb{R})$ is the function space of all functions $f : \mathbb{R} \to \mathbb{C}$ such that:
  - All orders of derivatives are continuous, i.e., they are in $C^\infty(\mathbb{R})$
  - The functions and their derivatives are rapidly decreasing.
- A function $f : \mathbb{R} \to \mathbb{C}$ is rapidly decreasing if it decrease faster than any polynomial:

$$\lim_{|x|\to\infty} |x^p f(x)| = 0, \; \forall p \in \mathbb{N}$$

- The space $\mathscr{D}(\mathbb{R})$ of test functions includes all the infinitely differentiable functions with bounded support.

$$\mathscr{D}(\mathbb{R}) = \{\varphi : \mathbb{R} \to \mathbb{C} | \varphi \in C^\infty(\mathbb{R}), \mathrm{supp}(\varphi) \text{ is bounded}\}$$

and, consequently, $\mathscr{D}(\mathbb{R}) \subset \mathscr{L}(\mathbb{R})$.

Spaces for aperiodic signals in DV

- In the case of discrete-variable, analogous spaces can be defined.
- In particular, spaces for absolutely and square summable signals such that:

$$\ell^1(\mathbb{Z}) = \left\{ x : \mathbb{Z} \to \mathbb{C} \text{ such that } x, \sum_{n=-\infty}^{\infty} |x(n)| < +\infty \right\},$$

$$\ell^2(\mathbb{Z}) = \left\{ x : \mathbb{Z} \to \mathbb{C} \text{ such that } x, \sum_{n=-\infty}^{\infty} |x(n)|^2 < +\infty \right\}.$$

- In most of textbooks, it is supposed that VD functions (i.e., sequences) are bounded and, therefore, $\ell^2(\mathbb{Z}) \subset \ell^1(\mathbb{Z})$.
- It is also possible to analogously define spaces for periodic sequences, $\ell_p(X_0)$:
  - Since the interval has a finite-length and the sequence values are bounded, any absolute or square summation will converge.