STATISTICAL METHODS FOR BUSINESS

UNIT 1: UNCERTAINTY AND PROBABILITY

UNIT 1. GOALS

- To understand the concepts of probability (classical, frequency, subjective).
- To be able to correctly apply the main rules of combinatorial calculus.
- To properly interpret the concepts of complementary event, union and intersection of events, mutually exclusive events and independent events.
- To identify partitions of the sample space and, upon the basis of a given partition, to apply the theorems of total probability and Bayes theorems.
- To properly interpret prior and posterior probabilities.

STATISTICAL METHODS FOR BUSINESS

UNIT 1: UNCERTAINTY AND PROBABILITY

1.1.- Probability. Concepts and measurement.

Motivation

Probability is a measure of uncertainty, that may be applied in many common situations

Will it rain tomorrow?

Can we expect an economic recession for next year?

Shall I win a lottery prize?

→Concept and meaning of probability

→Measurement

→Axiomatics

Meaning of probability

Classical Probability

$$P(A)=\frac{f.o.}{p.o.}$$

(f.o. = no. favourable outcomes).
(p.o. = no. possible outcomes).

How many possible outcomes?
Are all the outcomes equally likely?

Frequential Probability

$$fr(A)=\frac{n_A}{n}\rightarrow P(A)$$

Is it possible to repeat observations?
Do conditions remain constant?
Is it possible to make observations at all?

Subjective Probability

P(A) = 'Degree of Belief in A'

How do we assign probabilities?

Calculus of probabilities. Combinatorial numbers

What is the probability of winning a bet in a gamble with dice?
… And of winning a lottery prize?
… And of being selected for a jury in a trial?

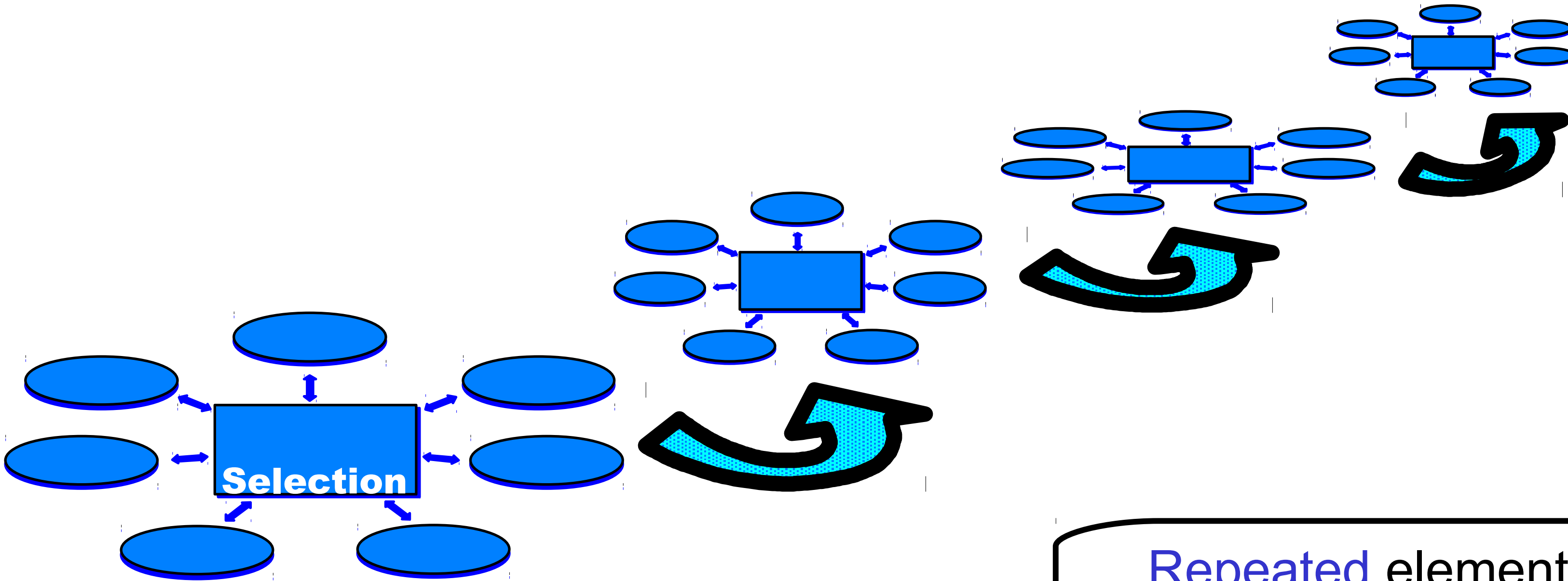
Variations with repetition

Variations

Permutations

Combinations

In how many ways can we draw n elements from a set of m elements?



Repeated elements are allowed

Variations with Repetition

$$VR_{m,n} = m.m\ldots m = m^n$$

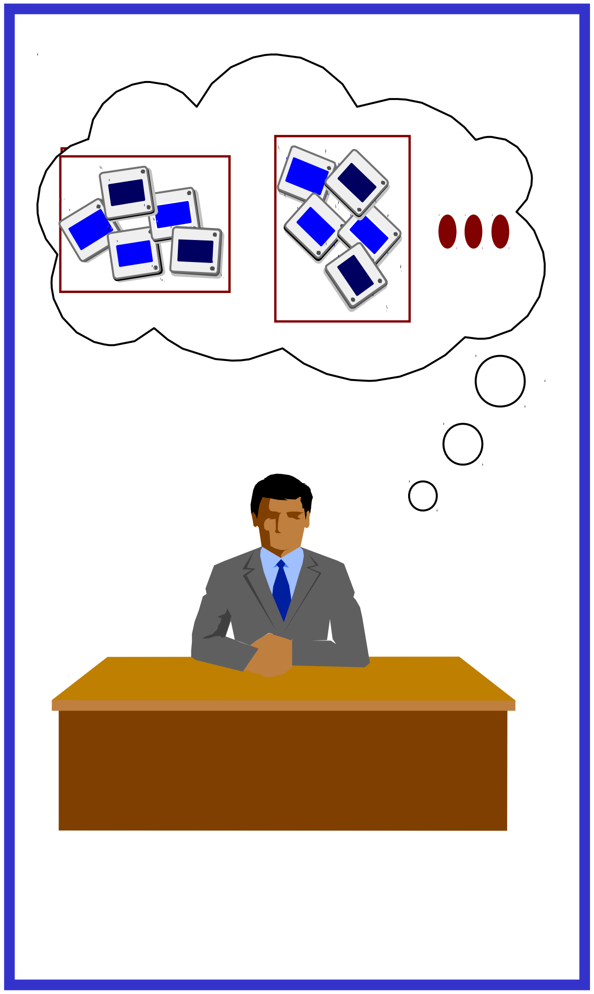
Order is relevant

Variations without Repetition

$$V_{m,n} = m.(m-1)\ldots(m-n+1)$$

Order is relevant

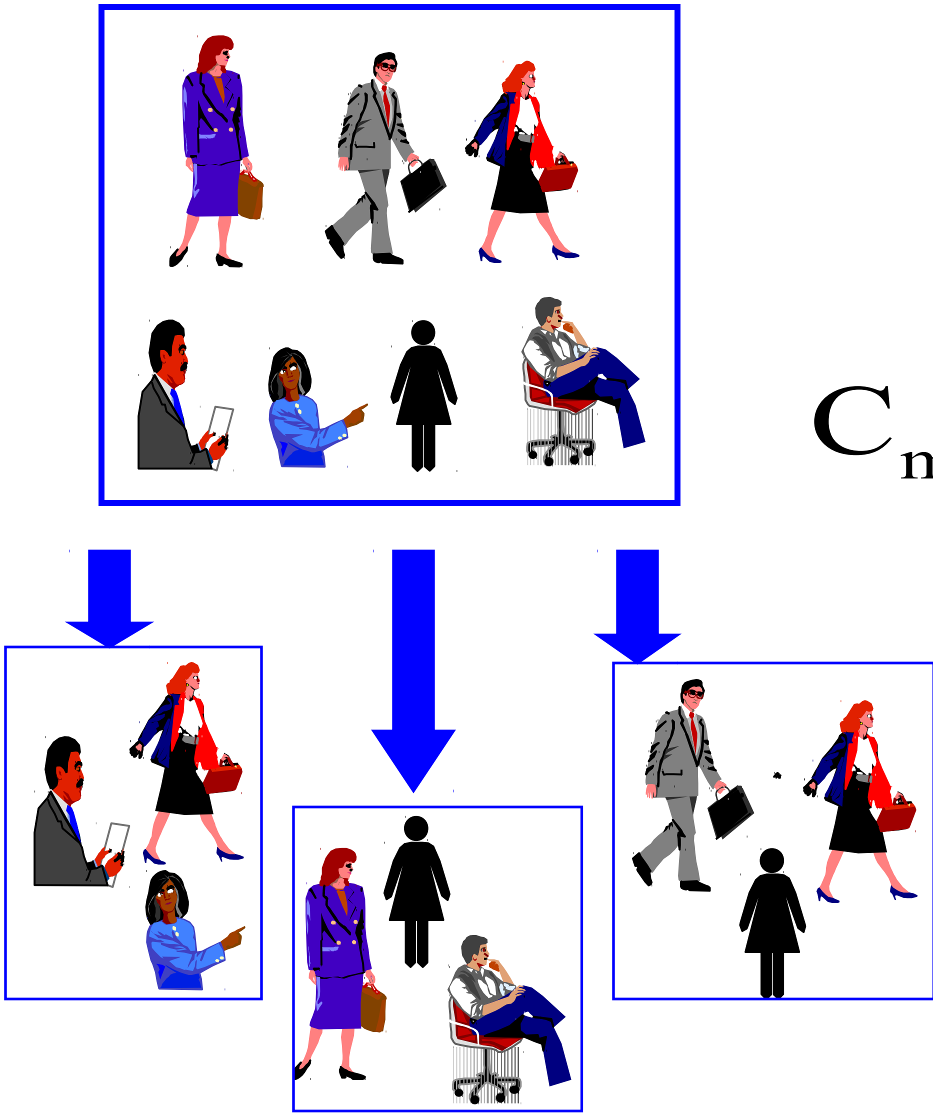
In how many ways can m elements be ordered?



Permutations

$$P_m = m!$$

How many subsets, each of size n, may be formed from a set of m elements?



Combinations

$$C_{m,n} = \binom{m}{n} = \frac{m!}{n!(m-n)!}$$

Order is NOT relevant

STATISTICAL METHODS FOR BUSINESS

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1.2.- Axiomatic definition of probability

Random experiments. Basic concepts

E **Sample Space** or *Sure Event* (= the set of all possible outcomes of a random experiment).

A, B ... **Events** (= subsets of E).

∅ **Null** (or impossible) **event**.

A∪B **Union** of events (= A, B, or both of them, happen).

A∩B **Intersection** of events (= both A and B happen).

A^c **Complement** of A (= A doesn't happen).

A∩B= ∅ **Mutually exclusive** events (= A and B cannot happen at the same time).

A collection **$\mathcal{A}$** of events E is called a **σ-algebra** if:

a) $\varnothing \in \mathcal{A}$.

b) $A \in \mathcal{A} \Rightarrow A^C \in \mathcal{A}$.

c) $A_1, \ldots, A_n, \ldots \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Illustration: A random experiment

Experiment: we toss a (balanced) die and observe the result (X).

A suitable sample space would be **E = {(X=1), (X=2), ..., (X=6)}**
(the set of all possible outcomes)

The outcome of the experiment is an element of **E, e.g., X=4.**

A σ-algebra of events for the experiment would be:

A = {∅, E, (X=1), (X=2), ..., (X=6), (X>2), (X ≤ 4), (0 < X ≤ 5),...} (e.g., the set of all subsets of E)

How do we compute the following probabilities?

$$P(X = 3),\ P(X \leq 2),\ P(X > 1),\ P(0 < X \leq 3)$$

Kolmogorov's Axiomatics

Definition. A probability measure is a mapping which assigns to every event $A \in \mathcal{A}$ a real number P(A), called the probability of A, so that the following axioms hold:

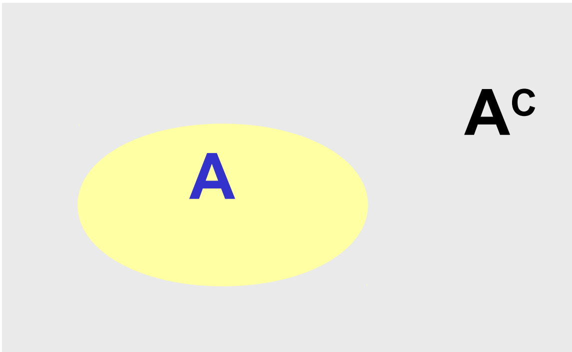
1) For any event $A \in \mathcal{A}$, it holds $P(A) \geq 0$.

2) $P(E) = 1$.

3) For any $A_1, \ldots, A_n, \ldots \in \mathcal{A}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$, it holds:

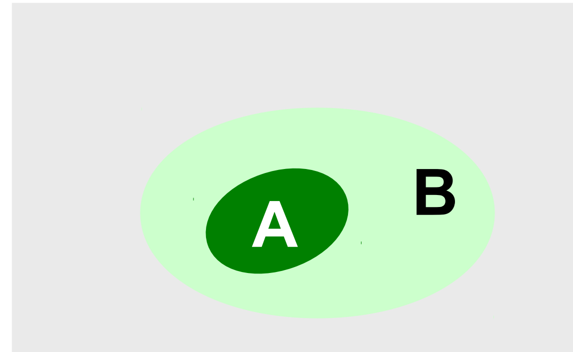
$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Some basic properties

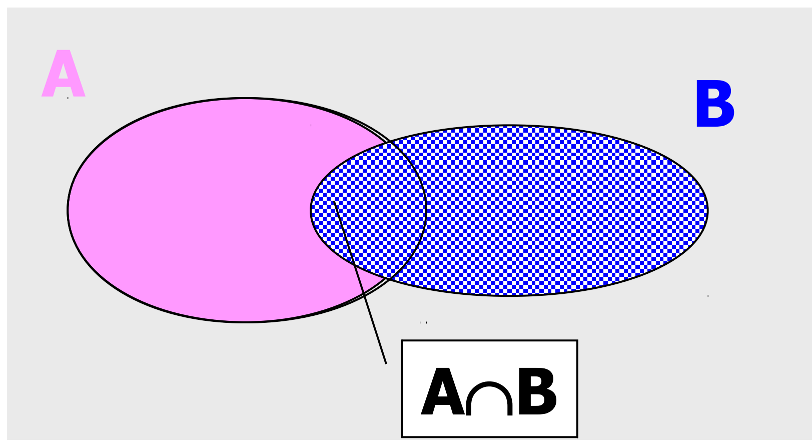


$$P(A^C) = 1 - P(A)$$

$$P(\varnothing) = 0$$



$$A \subset B \Rightarrow P(A) \leq P(B)$$



$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

STATISTICAL METHODS FOR BUSINESS

UNIT 1: UNCERTAINTY AND PROBABILITY

1.3.-Conditional probability and independence

Conditional Probability

Arrival of new information modifies the initial assignment of probabilities

A worker is at most 24 years old

What is the probability that he/she is unemployed?

AVAILABLE INFORMATION
B: 'age at most 24'

Probability of A (= being unemployed), given event B.

For any event A in **A**, the ***conditional probability*** of A given event B (provided P(B)>0), is defined as

$$P(\mathrm{A/B}) = \frac{P(A \cap B)}{P(B)}$$

Independence in probability

Is the probability of A affected by event B?

NO

A is independent of B

$\boxed{P(A/B) = P(A)}$ $\longleftrightarrow$ $P(A \cap B) = P(A)P(B)$

Condition of independence

A is independent of B <=> B is independent of A

Example

A survey on the labor status of 100 Spanish workers has provided the following results:

	Working (W)	Not working (N)
Asturias (A)	27	6
Basque Country (B)	59	8

$$P(A \cap N) = \frac{6}{100} = 0.06$$

Joint probability

$$P(N) = \frac{14}{100} = 0.14$$

Marginal probability

$$P(\mathrm{N/A}) = \frac{6}{33} = \frac{P(A \cap N)}{P(A)} = \frac{0.06}{0.33} = 0.182$$

$$P(\mathrm{N/B}) = \frac{8}{67} = \frac{P(B \cap N)}{P(B)} = \frac{0.08}{0.67} = 0.119$$

Conditional probabilities

Independence?

UNIT 1: UNCERTAINTY AND PROBABILITY

1.4.- Total Probability and Bayes Theorems

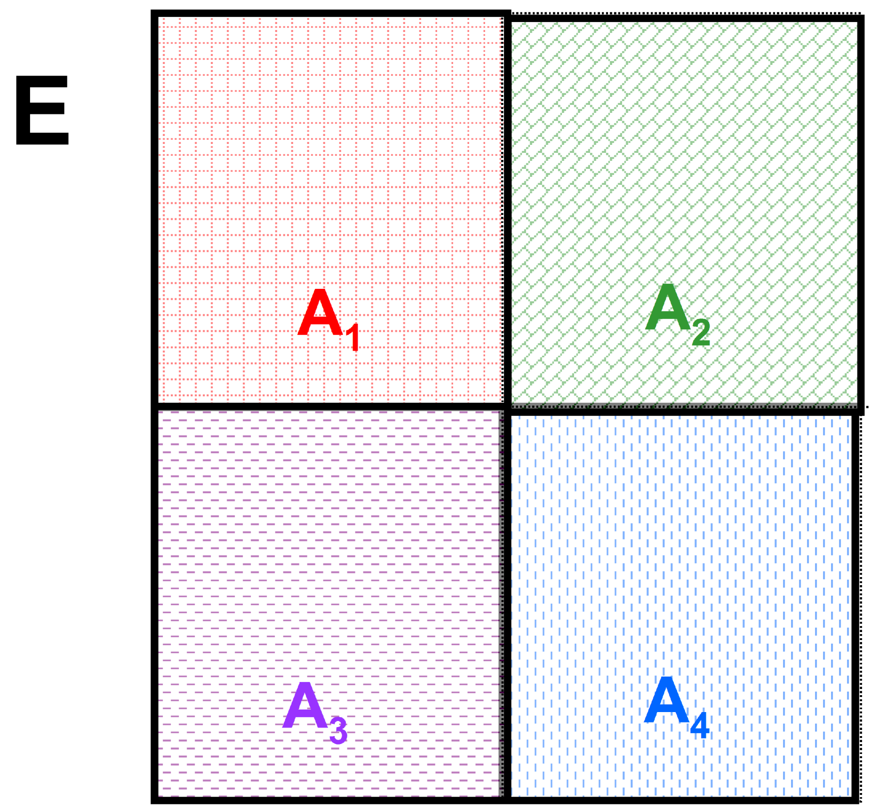
Partition or complete system of events

A collection of events $(A_1,\ldots,A_n)$ is called a partition of E (or a complete system of events on E) if the following conditions hold:

1) $P(A_i)>0 \quad \forall i=1,2,\ldots,n.$

2) $A_i \cap A_j = \emptyset \quad \forall i \neq j.$

3) $\cup_{i=1}^{n} A_i = E.$



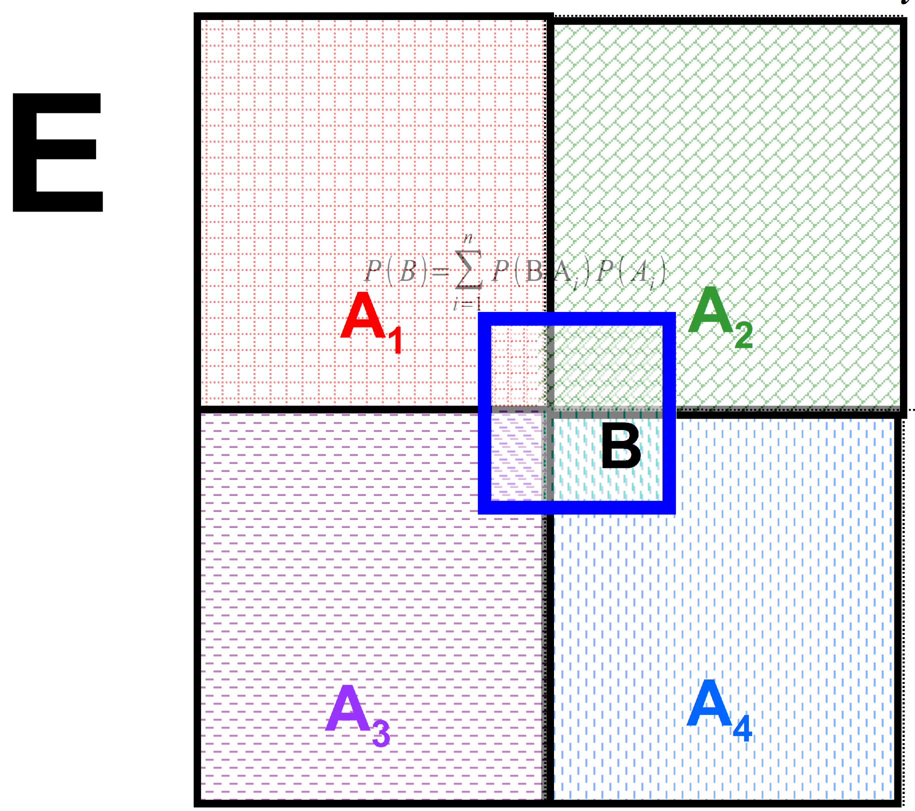
Theorem of Total Probability

For any partition (A_1, ..., A_n) defined on the probability space (E,**A**,P) and any event B in **A**, we have:

$$P(B)=\sum_{i=1}^{n} P(\mathrm{B}/\mathrm{A}_i)\,P(A_i)$$

Prior (or initial) probabilities

Likelihoods



Theorem of Total Probability. An illustration

The personnel of a firm are 70% engineers, 25% staff and 5% executives. In each of the above categories 40%, 30% and 70%, respectively, bought (B) shares of the firm.

Group	Prior Probabilities	Probability of having stocks (B)
Engineers (N)	0.70	0.4
Staff (S)	0.25	0.3
Executives (X)	0.05	0.7

Partition

Likelihoods

An employee of the firm is chosen at random. What is the probability that he/she owns shares of the firm?

$$P(B) = P(B/N)\,P(N) + P(B/S)\,P(S) + P(B/X)\,P(X) = 0.39$$

Bayes Theorem

For any partition (A_1, ..., A_n) of E and any event B, provided P(B)>0, it holds:

$$P\left(A_i/\mathrm{B}\right)=\frac{P\left(\mathrm{B}/\mathrm{A}_i\right)P\left(A_i\right)}{\sum_{i=1}^{n} P\left(\mathrm{B}/\mathrm{A}_i\right)P\left(A_i\right)}$$

Posterior (or final) probabilities

Bayes Theorem. An illustration

An employee has bought shares of the firm. What is the probability that he/she is an executive?

$$P\left(\mathrm{X/B}\right)=\frac{P\left(\mathrm{B/X}\right)P\left(X\right)}{P\left(B\right)}=\frac{0.70\times 0.05}{0.39}=0.09$$

Group	Prior Probability	Group probability given shares bought
Engineers (N)	0.7	0.72
Staff (S)	0.25	0.19
Executives (X)	0.05	0.09

Posterior probabilities