



Unit 5. Joint analysis and limit theorems

A. Class exercises

CARS The variables *number of family members* (X) and *number of cars per household* (Y) are analysed in a certain population, obtaining the following joint distribution:

Y \ X	1	2	3
1	0.19	0.16	0.09
2	0.06	0.19	0.31

- What is the probability that a household has two members and one car? Calculate the probability that a household has more than two members and less than four cars. What is the probability that a family has less than 3 members and one car?
- Obtain the marginal distribution of the number of cars per household. What is the proportion of households with more than two cars? Derive the marginal distribution of the number of family members.
- Calculate the vector of expectations and the covariance matrix for this population.
- Are X and Y independent?

Covariance Deduce the shortened formula of the covariance.

Ministry The ministry of economy of a certain country is analysing the relationship between *increase rate of exports* (X, percent) and *increase rate of GDP* (Y, percent). The following joint density function has been obtained:

$$f(x,y) = \frac{1}{3000}xy \quad \text{if } 5 < x < 20 \text{ and } 2 < y < 6,$$
$$f(x,y) = 0 \quad \text{otherwise.}$$

- What is the probability that exports grow over 8%?
- Calculate the expected GDP growth rate.
- Study the correlation between X and Y.

[From the text "*Análisis de Datos Económicos II. Métodos Inferenciales*", problem 4.1, pp. 214-216]

INCOME A car company has found that the daily sales of its two best selling models (respectively, X_1 and X_2 , both in thousand euros) are independent normally distributed RVs, with $X_1 \approx N(500, 30)$ and $X_2 \approx N(750, 40)$.

- What is the distribution of the variable "total daily sales of both models? Calculate the maximum total revenue obtained 40% of the days.
- What percentage of days does the revenue derived from Model 2 exceed that of Model 1 in at least 225,000 euros?



- c. If the sales of different days behave independently, calculate the probability that the average weekly sales exceed 1.27 million euros.

POST 5% of the packets received in a post office are returned undelivered.

- What is the probability that more than one package among the 5 received one day are returned?
- If 12 packets chosen at random (8 received one day, 4 another day) are observed: what is the probability that none of them is returned undelivered?
- In three different (randomly chosen) days 11, 9 and 12 packets, respectively, arrive at the office. Calculate the probability that the total number of packets returned in the three days exceeds 5.
- The monthly number of lost packets has a Poisson distribution with expectation 0.2. Calculate the probability that a certain quarter only 1 pack is lost.
- The weight of the packages is a random variable with mean 1.5 kg and standard deviation 0.5. If 10 packages are received one day, what is the probability that their total weight is ranging between 13 and 17 kg?

EMISSIONS The daily emissions of CO₂ (in tonnes) for a given company is a random variable with expectation 10 and standard deviation 2.

- Calculate the probability that the CO₂ emissions on a given day are between 6 and 14 tonnes.
- Under the assumption that the CO₂ emissions of different days are independent, calculate the probability that the total emissions made over a quarter (90 days) exceed 920 tonnes.
- The company will be fined by the environment ministry in the event that its quarterly emissions exceed 920 tonnes. What is the probability that the company is not fined in the next 7 quarters? Calculate the expected number of fines the company will suffer for exceeding the emission limit.

TELEPHONE The daily sales (in €) of each seller in a telephone company are a random variable with expectation 475 and standard deviation 30:

- Assuming that the sales of different days behave independently, what is the probability that in a 6-day week the total sales of a seller deviate from their expected value less than 100 €?
- Under the assumption of normality of daily sales:
 - Recalculate the probability requested in the above section. Compare with the previous result.
 - The manager of the company has decided to give a bonus to each seller whose sales exceed € 2,950 per week. If a sample of 34 sellers is chosen at random, what is the probability that less than 7 of them obtain the bonus on a certain week?



B. Class exercises

TV A study on household equipment is being conducted. The variables X (*number of children in the family*) and Y (*number of TVs at home*) have been analyzed, obtaining the following joint distribution:

$Y \setminus X$	1	2	3	4 or more
1	0.06	0.2	0.1	0.07
2	0.05	0.15	0.1	0.05
3	0.04	0.05	0.1	0.03

- Obtain the probability distribution of the number of TVs.
- What proportion of households have more than one TV?
- Are X and Y independent?
- What is the probability that a family has 2 children and more than 1 TV?

SOLUTION

a.

y_j	$p_{.j}$
1	0.43
2	0.35
3	0.22

b. 57%. c. Dependent. d. 0.2.

FISHING A study on the fisheries sector in a certain region has obtained that the average amount of fish landed daily is 16 tonnes, with a standard deviation of 3 tonnes.

- What is the probability that the amount of fish landed on a certain week deviates from its expected value by more than 30 tonnes?
- Calculate the probability that the total captures of 40 days exceed 600 metric tonnes.
- In the same study it was found that 35% of the ships land a small amount of fish. If on a certain day 20 ships come to port A and 14 ships arrive at port B, calculate the probability that at least a total of 13 ships have had a low level of captures.

SOLUTION

- a. At most 0.07. b. 0.98. c. 0.41.

SUPERMARKET The daily sales of food products (in million euros) in a chain of supermarkets are distributed $N(5, 1)$, whereas the daily sales of other products are normally distributed with expected value 2 million euros and standard deviation 0.5, with both RVs being independent.

- Complete the following statement: "30% of the days total sales do not exceed ... million. "
- What proportion of days the food product sales exceed those of other types of products by at least 2.5 million euros?
- If the total sales of 40 days are observed, calculate the probability that the average sales exceed 7.5 million euros (NOTE: assume that sales in different days are independent identically distributed).



SOLUTION

- a. 6.41 million. b. 67.36%. c. 0.0023.

BUS The time (in minutes) for an intercity bus in performing its usual route between two cities is a random variable with expectation 20 and standard deviation 6.

- If the bus makes 25 trips a day, obtain the probability that the average time deviates from its expectation by more than 4 minutes.
- Would the answer in the preceding paragraph change if the bus makes 32 journeys a day?
- The probability that there are no available seats on the bus (and, therefore, a client must wait for the next service) is 10%. If an individual uses the bus twice a day, what is the probability that in 20 days he/she has to wait for the next service less than 6 times?
- The monthly number of bus faults has a Poisson distribution with expectation 0.3. Calculate the probability that the bus suffers two breakdowns in a quarter.

SOLUTION

- a. 0.09 (at most). b. 0.0002 c. Below 0.7852. d. 0.1647.

BAKER'S The amounts of bread (X_1 , in thousand kilos) and pastries (X_2 , in thousands kilos) sold each month in a baker's shop are independent random variables, with $X_1 \sim N(20, 2)$ and $X_2 \sim N(22, 2.5)$.

- What is the probability that on a certain month the total sales of both products exceed 44,000 kilos?
- Those months when the combined sales of both products exceed 44,000 kilos the baker's employees receive a premium on their wages. What is the probability that the employees receive this bonus more than 12 times over the next three years?
- It is known that 20% of customers purchase both bread and pastries. Calculate the expected number of customers who arrive in the shop until someone buys both products.

SOLUTION

- a. 0.2646. b. 0.1292. c. 5.

MULTINATIONAL The annual benefits (X , in million euros) of each offshore branch of a multinational firm are normally distributed with expected value 0.2 and standard deviation 0.06.

- 70% of the branches with profits over 150,000 euros develop new investment plans in the future, while this proportion drops to 25% for those branches with profits under 150,000. Calculate the probability that a branch selected at random undertakes new investment plans.
- If the company has 40 branches in Europe, what is the probability that at least 25 of them carry out new investment plans?
- Employees receive an annual bonus of 300 euros in those branches with annual profits between 100,000 and 150,000 euros, or 600 euros if the annual profits of the branch exceed €150,000. Derive the distribution of the variable "annual bonus per employee".



SOLUTION

- a. 60.85%. b. 0.4801.
c. G: "annual bonus per employee".

$$P(G=0)=0,0475$$

$$P(G=300)=0,1558$$

$$P(G=600)=0,7967$$

SCORES The expected score for the exam of a college course is 6 points with standard deviation 1.1.

- If 4 friends do the exam, what is the probability that their average score is between 5 and 7 points?
- What is the probability that the average score of a group of 60 students exceeds 6.2 points?
- It is known through the college yearbooks that only 2% of students pass the course with honours ("matrícula de honor"). Consider the random variable X: 'number of students who pass with honours in a group of 60' and compute the probability that more than 2 students are in this category.

[From *Análisis de datos económicos II. Métodos inferenciales*, problem 4.3, pp. 217-218]

SOLUTION

- a. At least 0.6975. b. 0.0795. c. 0.1153.

FLORIST'S The daily sales (in euros) of a florist's shop are a random variable with expectation 500 and standard deviation 130.

- Assuming independence among the sales of different days, calculate the probability that the total sales of 7 days range between 3,000 and 4,000 euros.
- The owner intends to carry out some reforms in the shop. The cost of these reforms amounts to 42,000 euros, and it is expected that next quarter's sales will be enough to raise this sum. What is the probability that the total sales of the quarter may cover the cost of the reforms?
- Under the assumption of normality of the daily sales, recalculate a) and b) above. Compare with the previous results.
- According to available data, the proportion of customers who buy a plant is 10%:
d1) If 25 customers visit the shop one day, calculate the probability that at least 2 of them buy a plant.
d2) What is the expected number of customers entering the shop until someone requests a plant?

SOLUTION

- a. At least 0.5268. b. 0.9925. c. a) 0.853. b) 0.9925. d. d1) 0.7288. d2) 10.