Matemathics and Associated Teaching Methods I

THE NATURAL NUMBER

Degree in Primary Education Teaching

What is the number?

- It is a property of the sets.
- It is an **abstraction**, a **reflexive concept**.
  - Primary concepts: linked to contexts, to perceptions.

    Example: pipe

(Magritte, 1928)

  - Reflexive concepts: go beyond the context. They require a higher intellectual task, which is more complex.

Number mathematical construction

- A coordinability relation is defined on the set of all possible sets.
  - A and B are coordinable sets if there exists an bijective mapping between their elements.

- Each **natural number n** is the common property shared by all the sets belonging to each class made up by the coordinability relation.

Uses of the natural number

- To count: cardinal aspect
- To order: ordinal aspect
- To identify: nominal aspect
- To value
- To measure

Counting techniques

- They are based on the existence of an ordered words sequence (one, two, three,…) recited always in the same order.
- Ther are used to communicate information about:
  - the size of the sets (cardinal).
  - the place an element takes up in the set (ordinal)
  - the identification of an element in a set.

The principal counting principles

1. The abstraction principle
2. The stable-order principle
3. The order-irrelevance principle
4. The one-one principle
5. The cardinal principle

Verbal counting stages

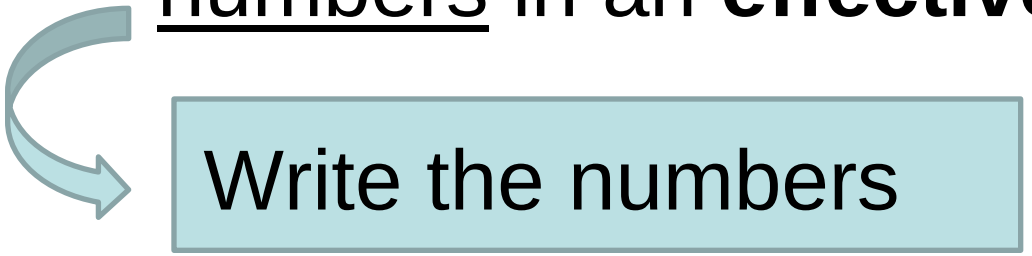
1. Memory sound repetition without sense (ordinal sense of the number)

2. Objects counting

3. Quantity of objects in a set (cardinal sense of the number)

Numeral systems

- They appear as a way for representing the numbers in an **effective** way.



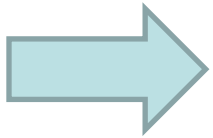
- They consist in a set of rules and agreements allowing to express in a verbal and graphical way **all the numbers** by means of words and symbols.

Primitive systems

- Quantities are represented by means of vertical marks.
- These marks are grouped together for large numbers.
- The system can be improved by using several equivalences.

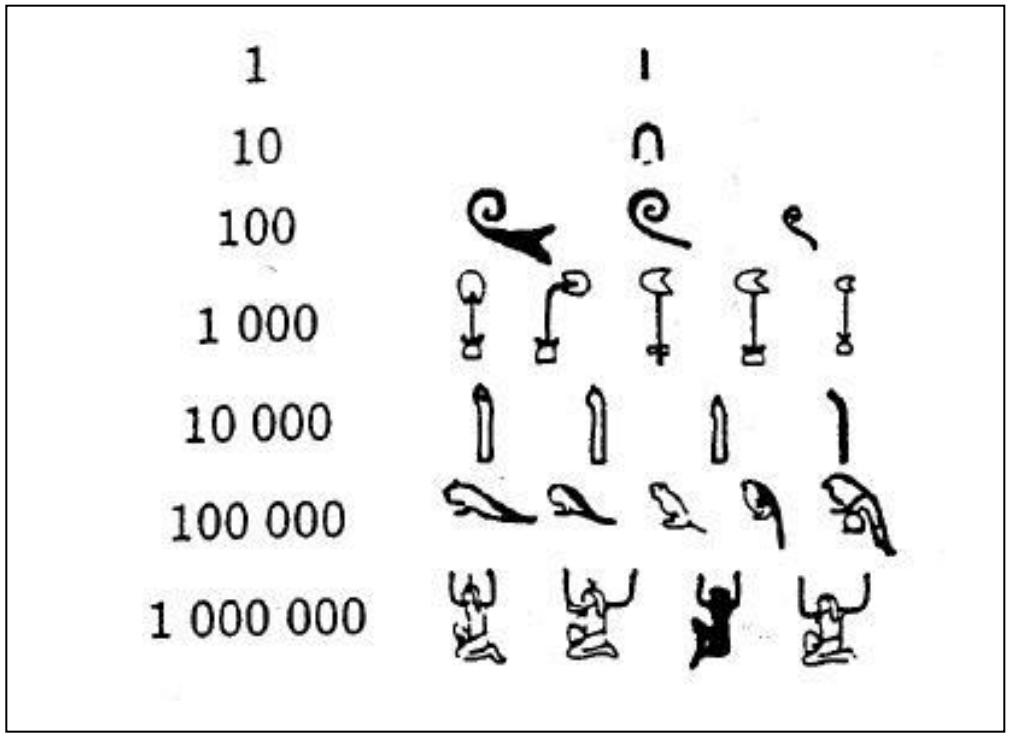
Additive systems

- Different symbols are created for certain quantities.
- The value of all symbols involved in the description of a global quantity is added.



They are little practical for representing large numbers.

Egyptian numeral system



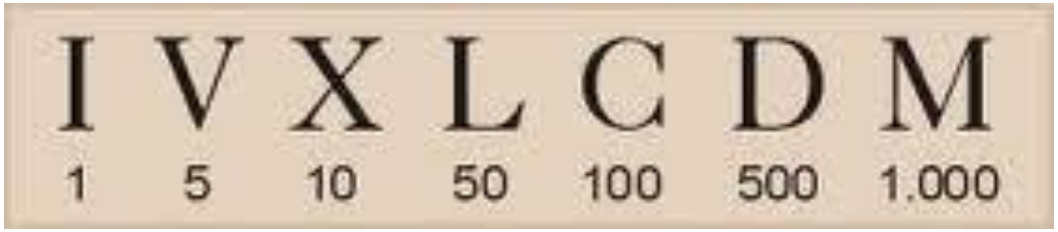
EXAMPLE:

200 000 3000 800

40 000 600 8

243 688

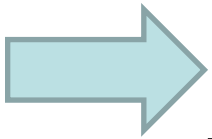
Roman numeral system



Chinese numeral system

Positional numeral systems

- Different symbols are created for the first numbers (until a “base” number).
- The value of each symbol depends on the position in which it is placed in the number representation.

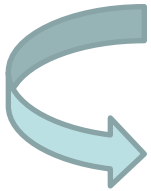


Very efficient systems, even for large numbers.

Positional representation systems

- Rules:
  - There are symbols from 1 to the base.
  - From the base: same symbols but in different positions.
- Important:

NUMBER ZERO IS NECESSARY FOR A PROPER FUNCTIONING OF ANY POSITIONAL SYSTEM



A symbol for zero is created

Example – base 6

In a beer factory, the small bottles are arranged in packs of 6; every 6 packs make up a package; every 6 packages make up a block; every 6 blocks make up a bundle, and every 6 bundles make up a pallet.

To simplify each order process the customers have to fill out the following form:

Pallets	**Bundles**	**Blocks**	**Packages**	**Packs**	**Small bottles**
…	…	…	…	…	…

Examples in other bases

- Base-60: first positional representation system -> **Babylonian system** (without symbol for zero).
- Base-2
- Base-4
- Base-12
- …

Hindu-arabic numeral system

- Origin: **India** (from 3rd century B.C.)
  - At the beginning: without symbol for zero.
  - Symbol 0: Hindu mathematicians (beginning of 6th century A.C.)
- **Arabian** conquest in the North India (11th century)

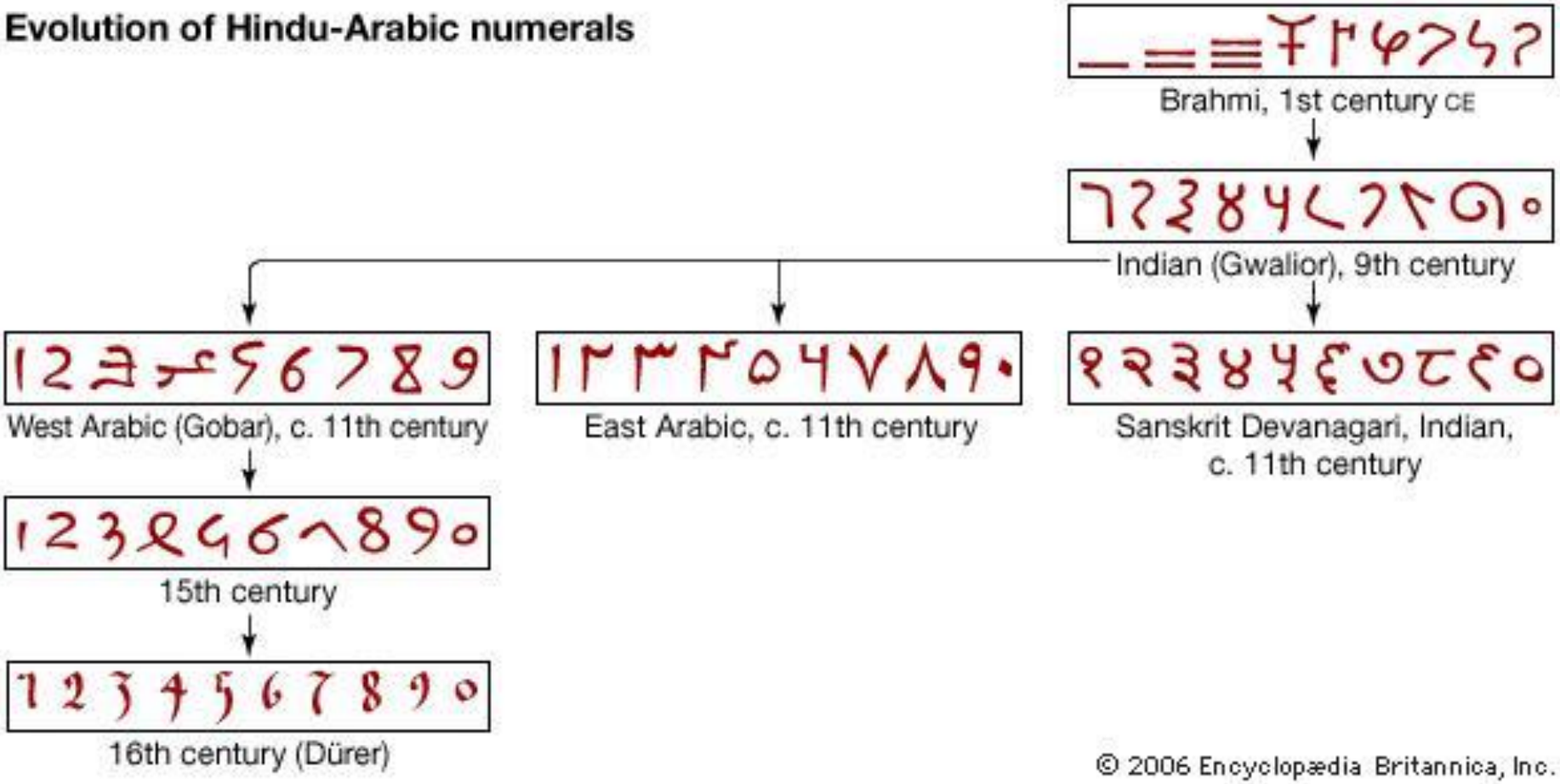
  → spreading to western Europe.
- Permanently instituted in Europe at the end of the 18th century.

Base-10 positional system (decimal system)

Ten symbols (digits)

- Ten units of a specific order make up a unit of an immediately higher order:
  - Ten units = one ten
  - Ten tens = one hundred
  - Ten hundreds = one thousand
  - Ten thousands = a ten thousand
  - …
- The order is related to the symbol position.

Graphical Hindu-Arabic symbols evolution



Operations with natural numbers

Mathematics learning according to Bruner's theory

- Bruner's stages of representation:

1. Enactive (or manipulative; action-based)
2. Iconic (image-based)
3. Symbolic (language-based: words and symbols)

Sum or addition

- To sum is to add up, to join together, to put together, to aggregate,…
- Mathematically, the sum of numbers corresponds to the union of disjoint sets.
- Operation result: addition – sum
- Components: addends

Basic properties

- Associative
- Commutative
- Neutral element
- Simplification

Difference or subtraction

- To subtract is to take off, to remove, to take away,…
- It is an operation bounded to the natural numbers order.
- Operation result: subtraction - difference
- Components: minuend and subtrahend

Subtraction definitions

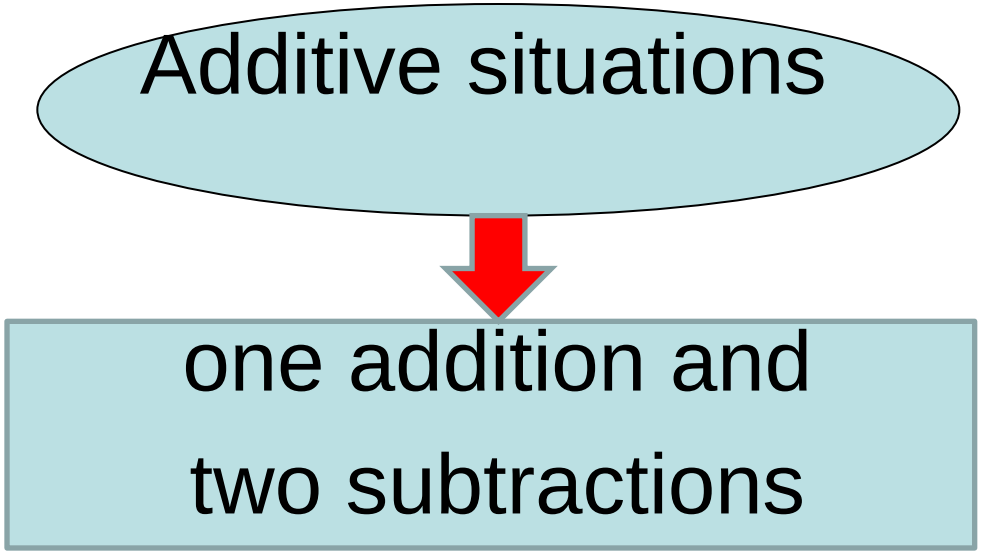
1. Set definition
2. Comparing set cardinals
3. Arithmetic definition

Basic properties

- It is not associative
- It is not commutative
- Neutral element
- Addition or subtraction of a quantity

ADDITION – SUBTRACTION
Conceptual field

- Conceptual field: set of situations whose treatment implies common concepts, procedures and symbol representations



Multiplication or product

- Repeated sum
- Operation result: multiplication – product
- Components: factors

Properties

- Associative
- Commutative
- Neutral element
- Distributive property of the product with respect to the addition/subtraction
- The distributive property of the addition/subtraction with respect to the product is not fulfilled
- Simplification

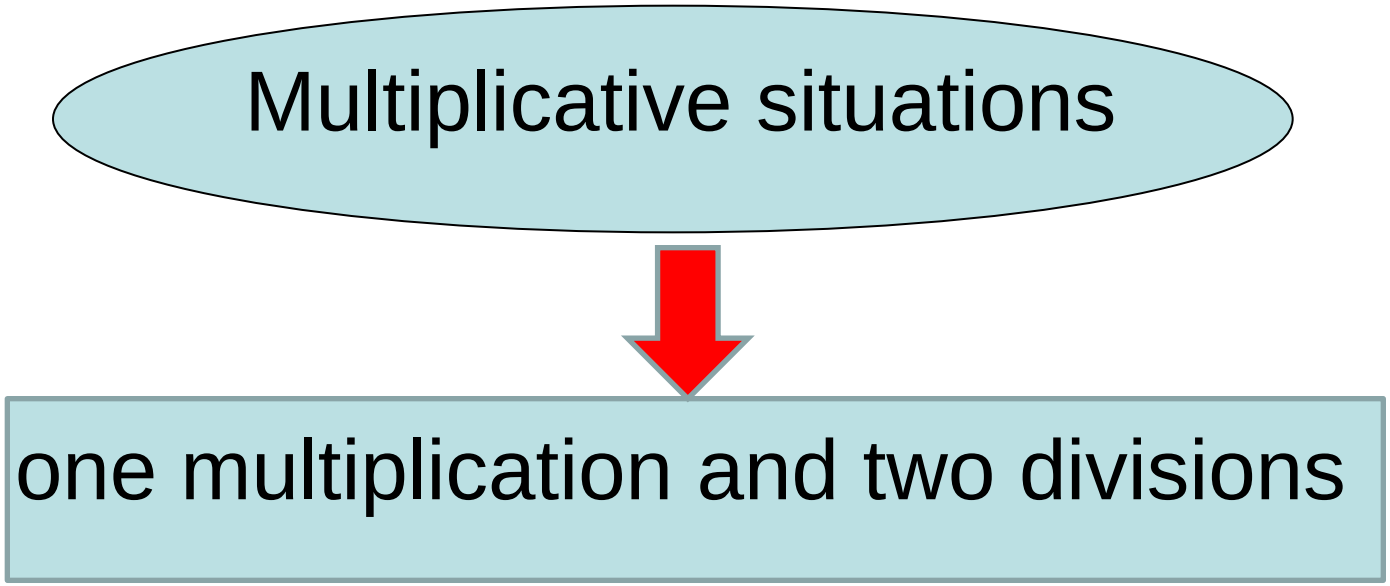
Division

- To divide is to share out, to distribute, to separate,… always **in equal parts**.
- Two different conceptions
- Set definition
- Fundamental division relation: division check formula
- **Exact** or **integer** division
- Operation result: division
- Components: dividend, divisor, quotient and remainder

Properties

- Dividing by zero is impossible, and it has no sense
- It is not commutative
- It is not associative
- Multiplication or division by a quantity ("crossing out zeros")

EXACT MULTIPLICATION – DIVISION
Conceptual field



Algorithms

- Etymology: *Al Kwaritzmi* (Persian mathematician, 9 A.C.)

- Definition: it is a finite series of rules applied in a specific order to a finite number of data in order to reach certain result in a finite number of stages, regardless of the data.

The algorithms corresponding to number operations are the result of a long historical process, which is constantly in progress.

abacus → algorithms → calculator → ?

Classical sum algorithm

- Basic rule: to sum units with units, tens with tens, hundreds with hundreds, and so on.

Position of the numbers

- Stages:

1. separately
2. in vertical
3. to improve
4. standarization

Classical subtraction algorithm

- Basic rule: to subtract units from units, tens from tens, and so on.

Position of the numbers

- Stages:

1. separately
2. standard

- Dificulty: **“carrying over”** → two methods:

1. “to-borrow algorithm” (natural process)
2. “to-ask-for-and-pay algorithm” (more artificial)

Classical multiplication algorithm

- Basic rules:
  - decomposition in units, cents, and so on
  - distributive property of the multiplication with respect to the addition
  - with more than two digits: associative property
- Stages:

1. separately
2. separately and vertically
3. vertically
4. standard

- Other algorithm: lattice method

Classicial division algorithm

- Very different characteristics:
  - Position of the numbers
  - It is carried out from left to right
  - We look for two results instead of one
  - Other algorithms are required
  - It is a semiautomatic algorithm: **decompose-estimate-check-redo**

Manipulative work

- Educational materials to make operations from a manipulative point of view:
  - Abacus
  - Rods
  - Multibase blocks
  - …