UNIT 3. DISCRETE PROBABILITY MODELS

UNIT 3. OBJECTIVES

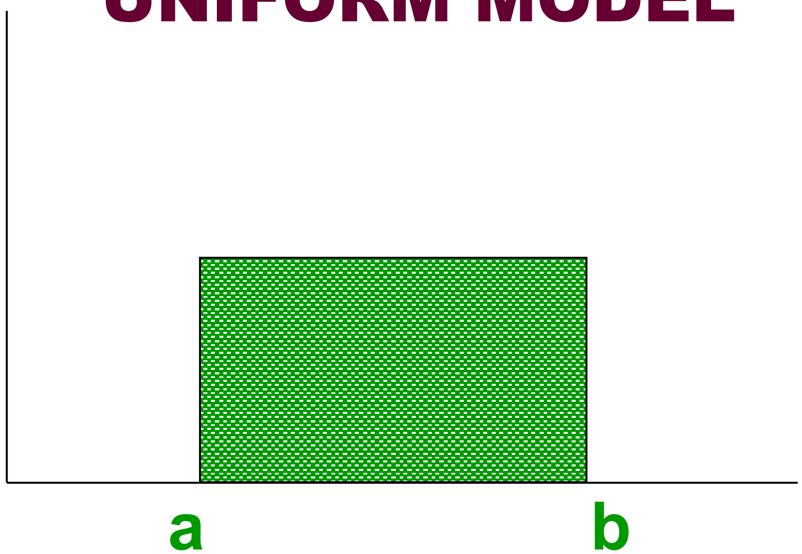
- To identify key discrete probability models, realizing the assumptions they are based on.
- To handle expressions of the expectation and the variance of the main models.
- To calculate probabilities for the main models.

PROBABILITY MODELS

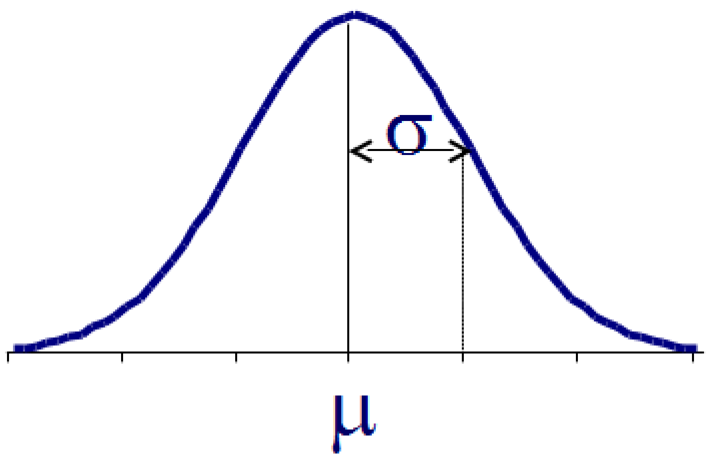
• The behaviour of different random variables may exhibit common patterns in practice.

• A **probability model** is a family of probability distributions indexed by a common set of parameters.

UNIFORM MODEL



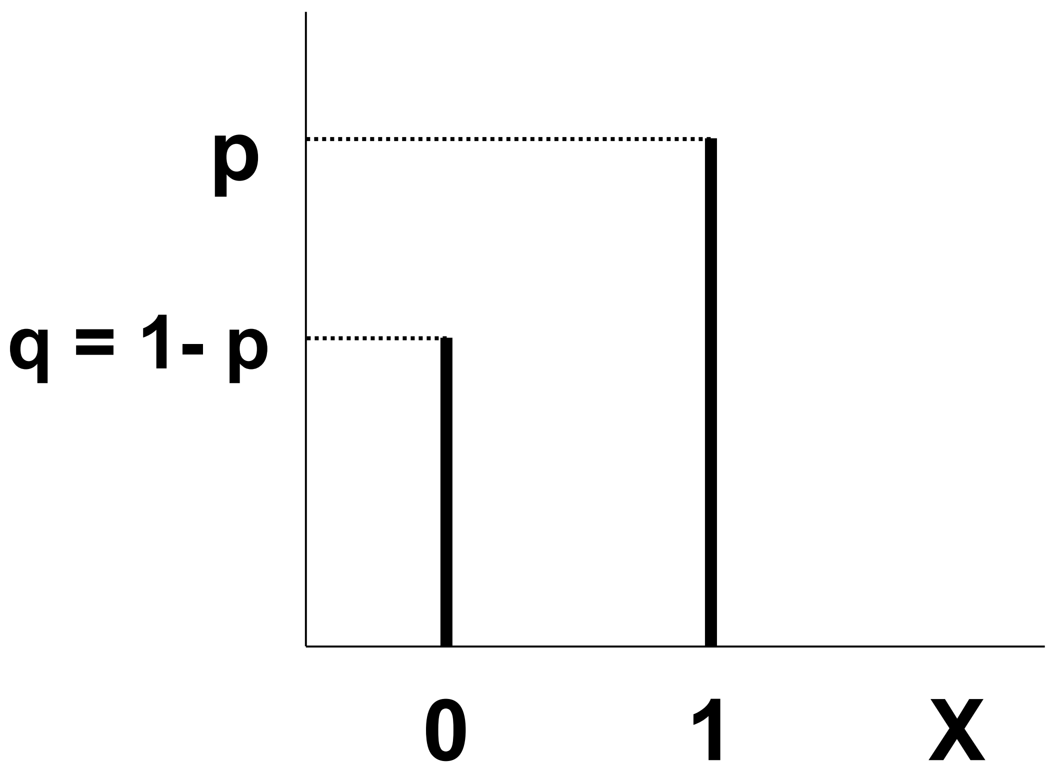
NORMAL MODEL



Bernoulli distribution

Random experiment	**Success**	**Failure**
Random variable	**1**	**0**
Probability	$P(X=1)=p$	$P(X=0)=q$

A SINGLE TRIAL



Characteristics:

$E(X)=p$

$Var(X)=pq$

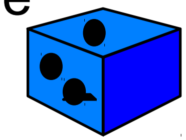
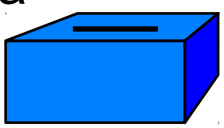
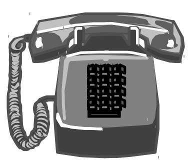
Bernoulli Process

Each trial is a random experiment with two possible outcomes

- **n independent trials are carried out.**
- **The n observations are mutually independient.**
- **The probability of success (p) is the same for all the trials.**

The assumptions of Bernoulli processes allow us to define several probability models: Binomial, Geometric, …

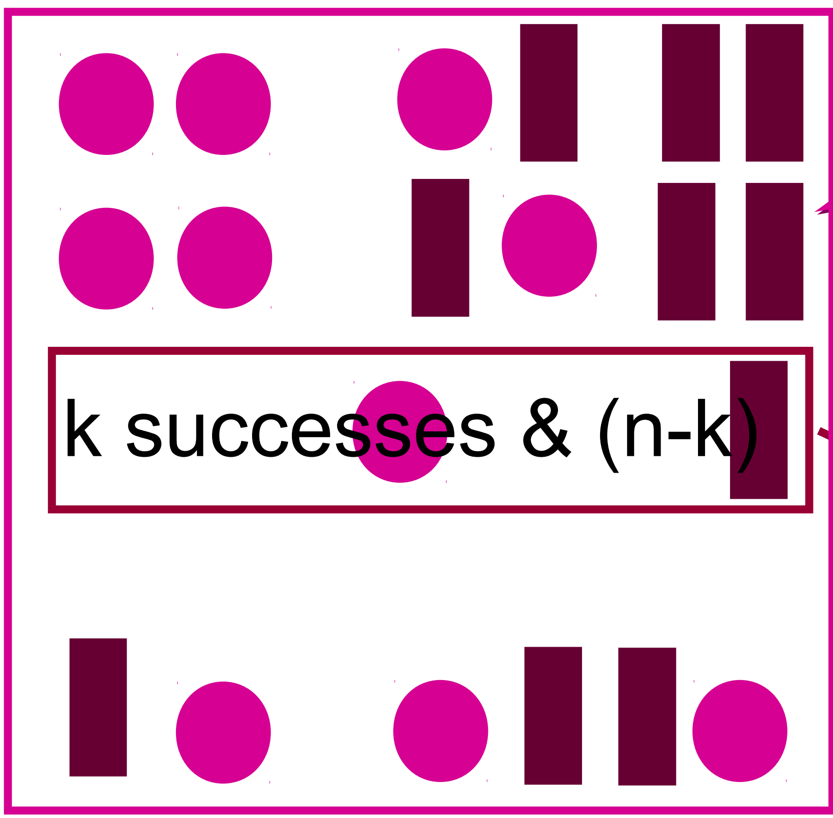
Bernoulli Processes

EXPERIMENT	SUCCESS	PROBABILITY OF SUCCESS	NUMBER OF TRIALS
Toss of a die 	**“To get a 2”**	$p = \frac{1}{6}$	10
Study on job status 	**“Employed”**	Employement rate	30
Poll before a referendum 	**“To vote YES”**	p=0.4	240
Call for an interview 	**“To attend”**	p	20

“Number of successes in n independent trials” ⟸ **B(n, p)**

Binomial model

SUCCESS	RV	Probability
"k successes in n trials"	$X \approx B(n, p)$	$P(X=k)$



No. Of sequences of n Trials with k successes

$$C_{n,k}$$

$$p^k (1-p)^{n-k}$$

● ="success"

▮ ="failure"

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Binomial distribution

X: Number of successes in n trials. $X \approx B(n,p)$

Probability function

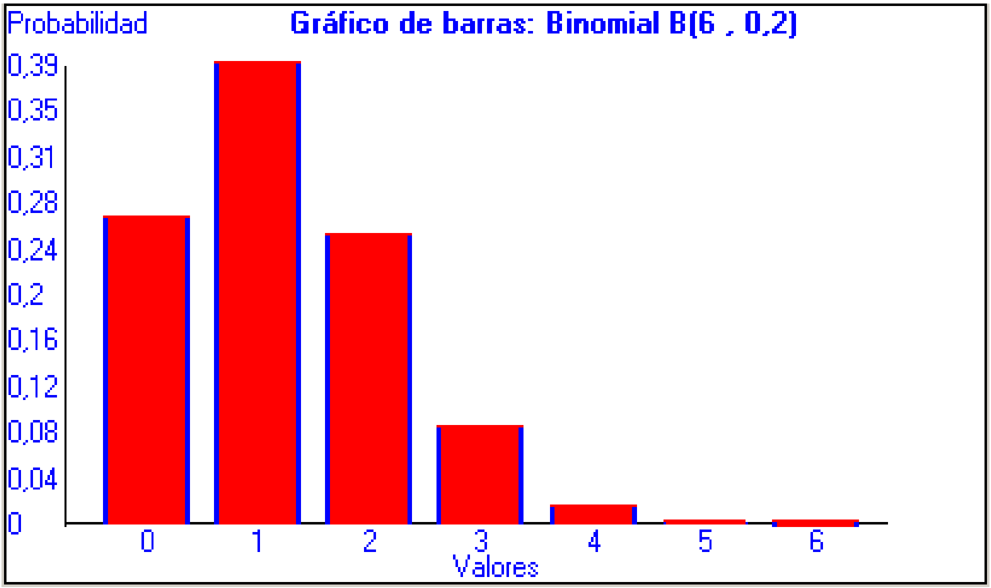
$$P(X=k)=\binom{n}{k}p^{k}(1-p)^{n-k}$$
$$k=0,1,\ldots,n$$

Characteristics:

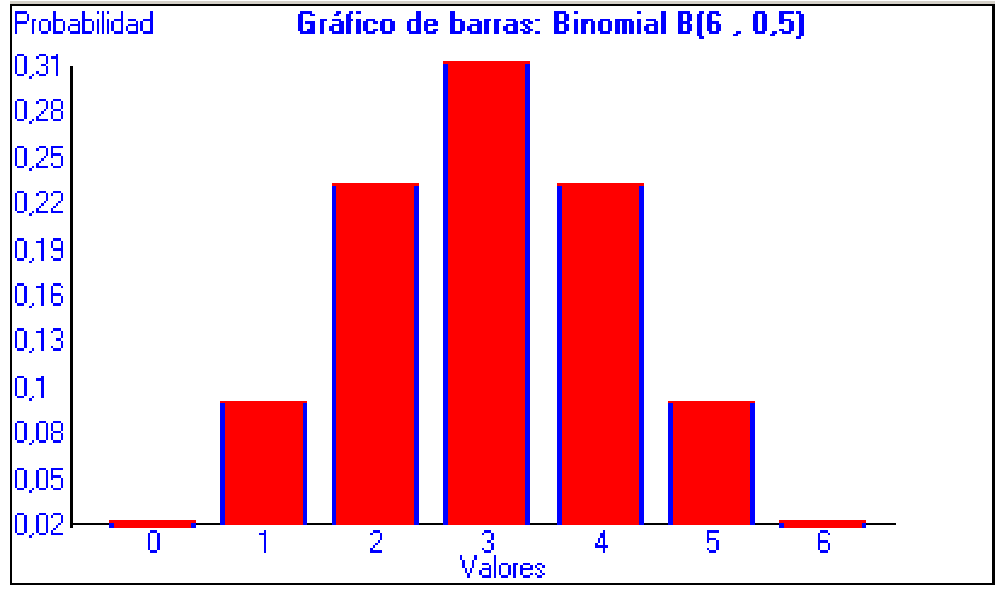
$$E(X)=np \qquad \mathrm{Var}(X)=npq$$

Some probability functions

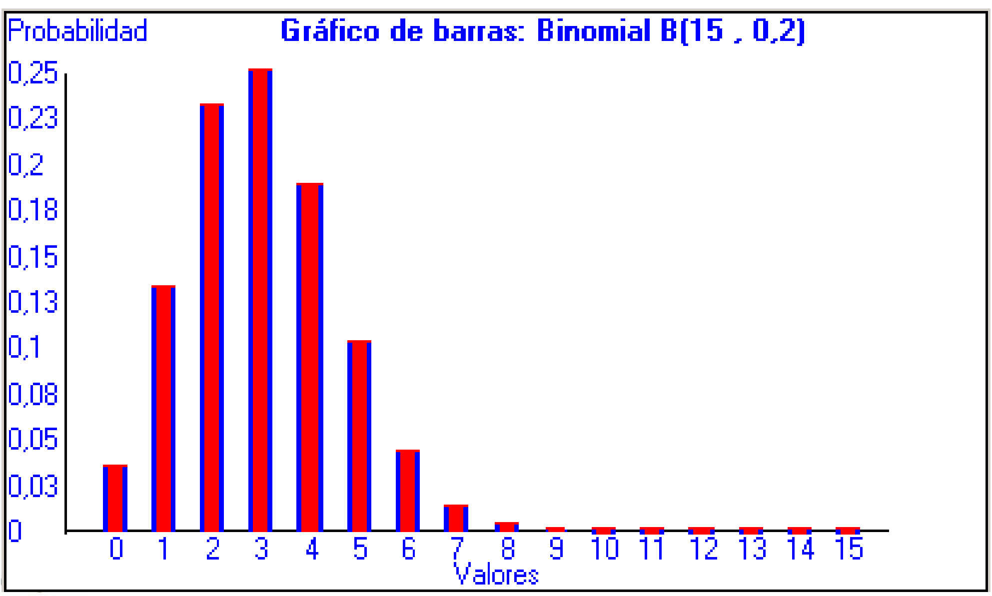
$X \approx B(n=6,\ p=0.2)$



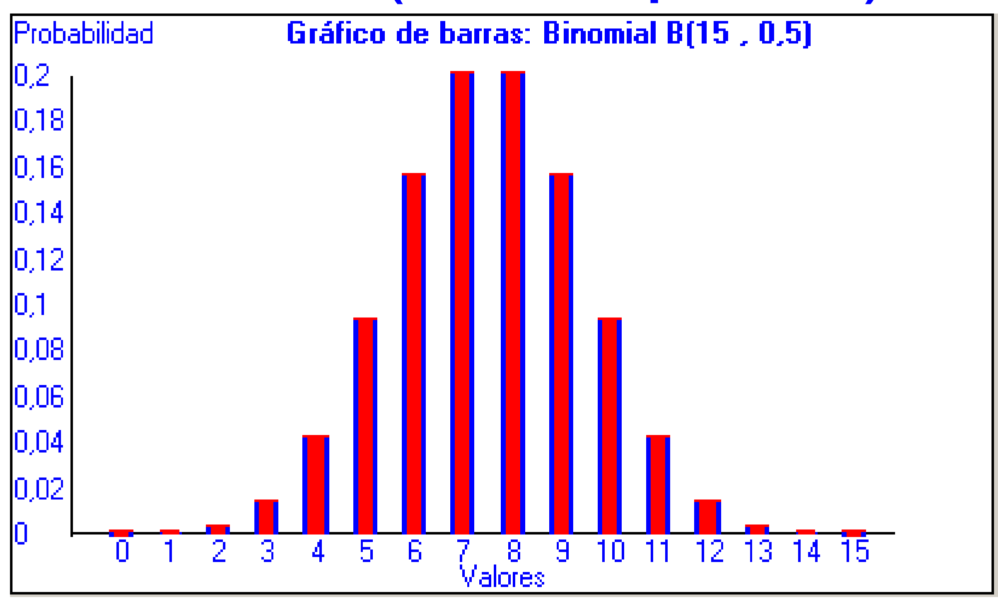
$X \approx B(n=6,\ p=0.5)$



$X \approx B(n=15,\ p=0.2)$



$X \approx B(n=15,\ p=0.5)$



Geometric model

EVENT
"No. of trials until the 1st. Success"

RV
$X \approx G(p)$

(k-1) times ■ and 1 ●

Probability
$P(X=k)$

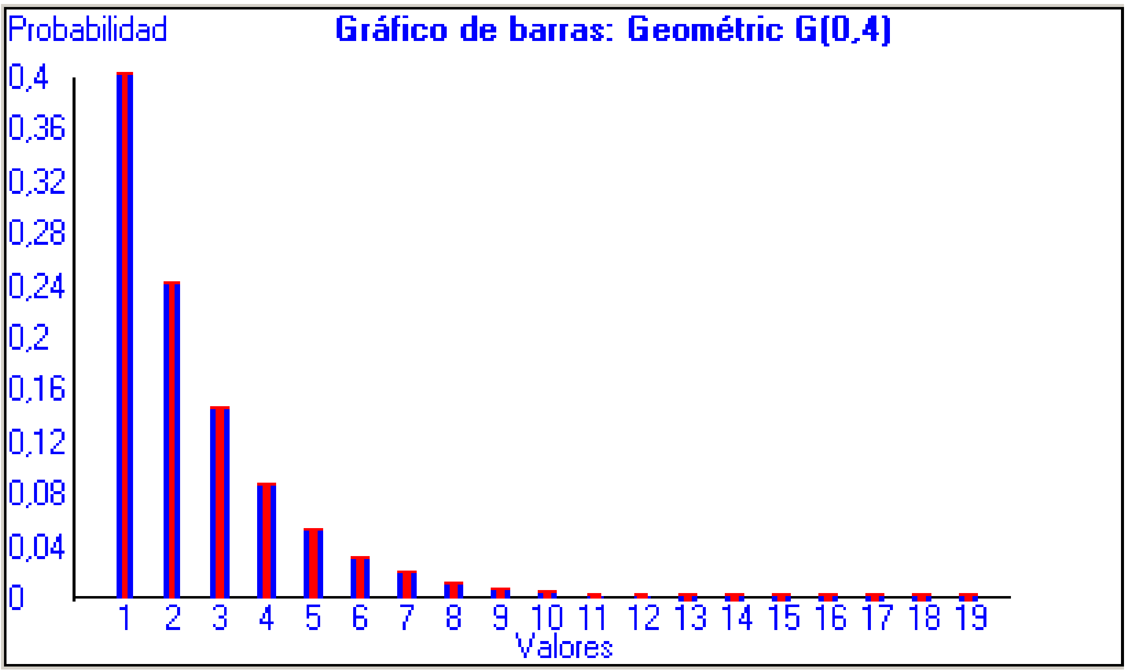
$$\boxed{(1-p)^{k-1}\,p}$$

● ="success"   ■ ="failure"

Geometric distribution

X: Number of trials until the first success. **X≈G(p)**

$$P(X=k)=(1-p)^{k-1}p \qquad k=1,\ldots,n,\ldots$$



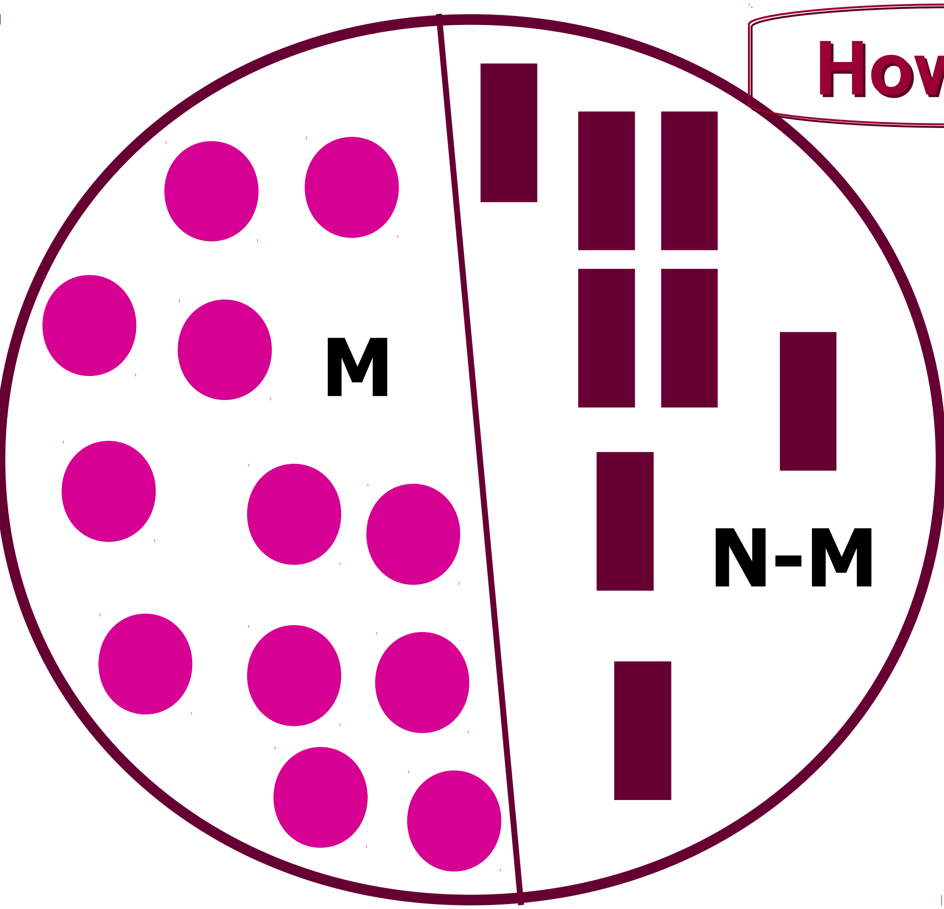
$$E(X)=\frac{1}{p}$$

$$Var(X)=\frac{q}{p^2}$$

Hypergeometric model

="sucess" ="failure"

X "Number of in the sample"

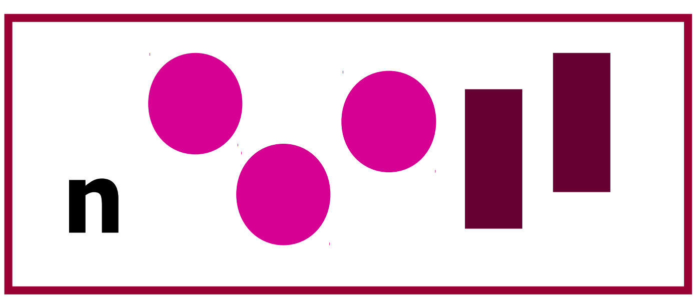


How many samples of size n?

$$C_{N,n}$$

How many samples of size n contain k success?

$$C_{M,k}\,C_{N-M,n-k}$$



$$P(X=k)=\frac{C_{M,k}\,C_{N-M,n-k}}{C_{N,n}}=\frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$$

SAMPLE (without replacement!)

Hypergeometric distribution

Probability function

$$P(X=k)=\frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$$

k= máx[0, n-(N-M)], …, mín(M,n)

Characteristics:

$$E(X)=n\frac{M}{N}=np$$

$$Var(X)=n\frac{M(N-M)(N-n)}{N^2(N-1)}=npq\boxed{\frac{(N-n)}{(N-1)}}$$

Correction factor

Convergence

$$H(N,M,n)\rightarrow B(n,p)$$

N > 50 and n <0.1N

Poisson model

- No. of people living more than 100 years.
- No. of plane accidents.
- No. of errata on a page.
- No. of telephones dialed incorrectly on a day.

Characteristics

- STABLE PROCESS (λ = average number of events per time unit)
- INDEPENDENT EVENTS
- PROBABILITY OF 2 OR MORE EVENTS IN A SHORT INTERVAL $\cong 0$

Poisson distribution

X "Number of events in a certain interval" $X \approx P(\lambda)$

Probability function

$$P(X=k)=\frac{e^{-\lambda}\lambda^{k}}{k!}$$

$k = 0, 1, 2, \ldots, n, \ldots$

Characteristics

$E(X) = \lambda$

$Var(X) = \lambda$